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# Large volume susy breaking with a solution to the decompactification problem

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## Abstract

We study heterotic ground states in which supersymmetry is broken by coupling the momentum and winding charges of two large extra dimensions to the R-charges of the supersymmetry generators. The large dimensions give rise to towers of heavy string thresholds that contribute to the running of the gauge couplings. In the general case, these contributions are proportional to the volume of the two large dimensions and invalidate the perturbative string expansion. The problem is evaded if the susy breaking sectors arise as a spontaneously broken phase of  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  supersymmetry, provided that  $\mathcal{N} = 4$  supersymmetry is restored on the boundary of the moduli space. We discuss the mechanism in the case of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  orbifolds, which requires that the twisted sector that contains the large extra dimensions has no fixed points. We analyze the full string partition function and show that the twisted sectors distribute themselves in non-aligned  $\mathcal{N} = 2$  orbits, hence preserving the solution to the string decompactification problem. Remarkably, we find that the contribution to the vacuum energy from the  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  sectors is suppressed, and the only substantial contribution arises from the breaking of the  $\mathcal{N} = 4$  sector to  $\mathcal{N} = 0$ .

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## 1. Introduction

String theory is the leading contender for a unified theory of all known interactions [1], and numerous string models exhibiting rich phenomenological properties have been constructed. They utilize various compactification techniques, like for instance the Calabi–Yau compactifications [1], the orbifold compactifications [2], the 2d-fermionic constructions [3], the self-dual lattice constructions [4], the asymmetric orbifold compactifications [5], the  $\mathcal{N} = (2, 2)$  superconformal constructions [6], as well as the  $\mathcal{N} = (2, 0)$  constructions [3,5].

However, all of the quasi-realistic string models that have been constructed to date, namely with the correct standard model spectrum, possess an  $\mathcal{N} = 1$  spacetime supersymmetry (susy), and the question of how this symmetry is broken is still an open problem. The mechanisms that have been proposed to address this point are either perturbative [7–10] or non-perturbative [11–14]. One can consider:

- A non-perturbative breaking *via* gaugino condensation [11], which up till now has to be discussed at the level of the effective supergravity. Due to the non-perturbative nature of the mechanism, one loses the predictability associated to the underlying string model. One then has to resort to an effective parametrization of the susy breaking parameters.
- Perturbative and/or non-perturbative flux compactifications, where internal fluxes are introduced and break susy suitably. These models can be explored using the non-perturbative  $S, T, U$ -dualities between the heterotic, Type IIA, Type IIB and orientifold superstring vacua [13–15].
- An interesting example of geometrical fluxes is the one associated with a Stringy Scherk–Schwarz (SSS) susy breaking compactification, which has the advantage to be implemented at the perturbative string level [9]. Here, the symmetry breaking parameters are obtained directly from the perturbative string theory.

In this last approach, the Scherk–Schwarz mechanism [16] defined in supergravity theories is promoted at the superstring level [8–10]. Denoting the string scale as  $M_s = 1/\sqrt{\alpha'}$ , the mechanism entails that some of the compactified dimensions of characteristic size  $R/M_s$  (measured in string frame) of the internal manifold are large, *i.e.* of the order of the inverse of the supersymmetry breaking scale. In Einstein frame, we have  $m_{\frac{3}{2}}^{(E)} = \mathcal{O}(M_{\text{Planck}}/R) = \mathcal{O}(1\text{--}10)$  TeV. This follows from the fact that supersymmetry is broken by coupling a  $\mathbb{Z}_2$  freely acting shift in these compactified directions, with the R-charges of the supersymmetry generators. These large dimensions give rise to tower of states, charged under low-energy gauge groups, that populate the energy range between the susy breaking scale and the Planck scale. They induce thresholds, whose analysis was recently pioneered in [17], that contribute to the running of the gauge couplings, Yukawa couplings and soft susy breaking parameters.

However, a problem arises when the threshold corrections are proportional to the volume of the large dimensions. When the  $\beta$ -function coefficient is negative, they drive the theory to strong coupling at energies lower than the unification (or string) scale [18]. This problem is known as the *decompactification problem* and some proposals exist on how to avoid it [9,10,15,18]. A first idea supposes the existence of models without  $\mathcal{N} = 2$  sectors, so that the threshold corrections are independent of the volume moduli of the internal theory [10]. Alternatively, one can suppose the thresholds of different spin states cancel among each other at one-loop in the perturbative expansion [10]. However, the stability of this mechanism against higher loop corrections has not

been demonstrated. Moreover, no quasi-realistic model realizing one of the above two proposals has been constructed so far.

In this paper, we examine a different possibility, which was introduced in Ref. [18] in the context of  $\mathcal{N} = 2$  supersymmetric models. Due to the properties of the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  spontaneous breaking *via* freely acting orbifolds, the behavior of thresholds as functions of the moduli of the internal manifold is radically different from that of the generic orbifold models, where the breaking from  $\mathcal{N} = 4$  to  $\mathcal{N} = 2$  is not spontaneous [18]. The reason for this distinction is that  $\mathcal{N} = 4$  supersymmetry is restored on the boundary of the moduli space. In this case, for large values of the relevant moduli, the thresholds vanish (up to logarithmic corrections).

In order to extend the above idea to non-supersymmetric models, we first present in Section 2 the class of string theories we consider, namely the heterotic  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-left/right-symmetric orbifold models realized in “moduli-deformed fermionic constructions”, where the  $\mathcal{N} = 1$  supersymmetry is further spontaneously broken to  $\mathcal{N} = 0$  by a SSS mechanism. In Section 3, we provide some preliminary introduction on how the gauge coupling threshold corrections in simple  $\mathcal{N} = 4$  models spontaneously broken to  $\mathcal{N} = 0$  do not develop dangerous linear dependences on volume moduli. We turn back to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models from Section 4 to the end of the article. For simplicity, we specialize to the case where only one of the  $\mathbb{Z}_2$  actions is freely acting. The second *together with* the diagonal action of the two are supposed to have fixed points. As we will see, this restriction forces the spontaneous breaking of the supersymmetries to involve only one of the three internal 2-tori, for the decompactification problem not to arise.

In Section 4, we evaluate the threshold corrections and effective potential generated at one-loop in the sectors arising from the action of a single  $\mathbb{Z}_2$ , namely the  $\mathcal{N} = 4$  sector and the so-called  $\mathcal{N} = 2$  1st complex plane. For the associated  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  susy breaking to be spontaneous, the  $\mathbb{Z}_2$  twist acts simultaneously as a shift along some of the two untwisted internal directions. The SSS mechanism responsible of the final spontaneous susy breaking to  $\mathcal{N} = 0$  is implemented by an additional  $\mathbb{Z}_2^{\text{shift}}$ . The action of the latter on the above two untwisted internal directions introduces sub-sectors we analyze carefully. We find that only the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  sub-sector (denoted as *B*), together with two sub-sectors (denoted as *C* and *D*) preserving distinct  $\mathcal{N} = 2$  supersymmetries contribute substantially.

Section 5 discusses physically the formal results obtained in the sub-sectors *B*, *C*, *D*. Three moduli-dependent mass scales  $M_{B,C,D}^{(E)}$  are introduced, the lowest of which being in the TeV region in realistic models. These scales, which are different from the gravitini masses present in each sector, control the contributions of the whole towers of Kaluza–Klein states that contribute to the running effective gauge couplings. Some examples are also presented.

Section 6 completes the sector by sector analysis of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models, by considering the additional contributions arising from the action of the second  $\mathbb{Z}_2$ , namely the 2nd and 3rd complex planes, together with the  $\mathcal{N} = 1$  sector. Under our hypothesis (only the 1st  $\mathbb{Z}_2$  action is freely acting), the above two planes have fixed points and the SSS susy breaking to  $\mathcal{N} = 0$  must only involve the 1st plane moduli. This has two consequences. First, the gravitino mass  $m_{\frac{3}{2}}$  of the  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  model is of order  $1/\sqrt{\text{Im } T_1}$ , the inverse of the volume of the internal 1st plane. Moreover, the 2nd plane, 3rd plane and  $\mathcal{N} = 1$  sectors preserve exact supersymmetries at tree level and the threshold scales  $M_I^{(E)}$  associated to the complex planes  $I = 2, 3$  must be of the order of the Planck scale. We also collect our results in order to write the expression of the effective coupling constants in the  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  models we consider. Moreover, it is remarkable that the effective potential arises only from the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  sector *B*, the other

sectors being either supersymmetric or exponentially suppressed, when  $m_{\frac{3}{2}}^{(E)}$  is lower than the Planck scale.

Finally, our conclusions can be found in Section 7, while Appendix A is a review of the moduli-deformed fermionic construction.

## 2. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ models with spontaneously broken susy

The context in which we will propose a solution to the decompactification problem consists in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifolds obtained *via* the “moduli-deformed fermionic construction” defined in Appendix A, and describing a spontaneous  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  susy breaking. As we will see in Section 4, the relevant models rely on an underlying  $\mathcal{N} = 4$  structure. Specifically, at least one of the two  $\mathbb{Z}_2$ ’s must act freely, so that an  $\mathcal{N} = 2$  sector will have the desired properties of spontaneously broken  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  [18]. It is however important to note that this condition is incompatible with the existence of a chiral spectrum, as explained in Section 6. The final implementation of the  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  spontaneous breaking is done by coupling another  $\mathbb{Z}_2$  freely acting shift in the large internal directions, with the supersymmetric R-symmetry charges (*e.g.* the four  $SO(1, 9)$  helicity charges of the ten dimensional mother theory). In the present section, our goal is to review the expression of the gauge threshold corrections in heterotic string and to present the structure of the partition function in the *most general*  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifold models arising from deformed fermionic construction.

For a gauge group factor  $G^i$  at Kac–Moody level  $k^i$ , the running effective field theory coupling constant of a string model is [15,18–21]

$$\frac{16\pi^2}{g_i^2(\mu)} = k^i \frac{16\pi^2}{g_s^2} + b^i \log \frac{M_s^2}{\mu^2} + \Delta^i, \quad (2.1)$$

where  $b^i$  is the  $\beta$ -function coefficient,  $g_s$  is the string coupling and  $\mu$  plays the role of renormalization scale in the effective field theory. In string calculations, a mass gap  $\mu$  is introduced to regularize the infrared [20]. The analytic expression of the threshold corrections takes the form

$$\begin{aligned} \Delta^i = & \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left( \frac{1}{2} \sum_{a,b} \mathcal{Q}_b^{[a]}(2v) \left( \mathcal{P}_i^2(2\bar{w}) - \frac{k^i}{4\pi\tau_2} \right) \tau_2 Z_b^{[a]}(2v, 2\bar{w}) - b^i \right) \Big|_{v=\bar{w}=0} \\ & + b^i \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}}, \end{aligned} \quad (2.2)$$

where  $Z_b^{[a]}(2v, 2\bar{w})$  is the partition function for given spin structures  $(a, b)$  of the world-sheet fermionic supercoordinates.  $(a, b)$  are integer modulo 2: Spacetime bosons have  $a = 0$ , while spacetime fermions have  $a = 1$ . As indicated by the presence of the variables  $v$  and  $\bar{w}$ ,  $Z_b^{[a]}(2v, 2\bar{w})$  is actually a refined partition function, on which the helicity operator  $\mathcal{Q}_b^{[a]}(2v)$  acts on the left-moving part,

$$\mathcal{Q}_b^{[a]}(2v) = \frac{i}{\pi} \partial_\tau \left( \log \frac{\theta_b^{[a]}(2v)}{\eta} \right) \equiv \frac{1}{16\pi^2} \frac{\partial_v^2(\theta_b^{[a]}(2v))}{\theta_b^{[a]}(2v)} - \frac{i}{\pi} \partial_\tau \log \eta. \quad (2.3)$$

Our conventions for the  $\theta_b^{[\alpha]}(v|\tau)$ -functions can be found in Eq. (A.1) or in Appendix C of Ref. [22] and it is understood that  $\theta_b^{[\alpha]}(v)$  denotes  $\theta_b^{[\alpha]}(v|\tau)$ , while  $\theta[\alpha_\beta]$  stands for  $\theta_b^{[\alpha]}(0|\tau)$ . On the contrary,  $\mathcal{P}_i(2\bar{w})$  is the charge operator of the gauge group factor  $G^i$ , thus acting on the

right-moving sector of the heterotic string as a derivative operator. Finally, no infrared divergence occurs in the expression of  $\Delta^i$ , due to the relation

$$b^i = \lim_{\tau_2 \rightarrow \infty} \frac{1}{2} \sum_{a,b} \mathcal{Q}_b^a(2v) \mathcal{P}_i^2(2\bar{w}) \tau_2 Z_b^a(2v, 2\bar{w}) \Big|_{v=\bar{w}=0}. \quad (2.4)$$

In all orbifold models that preserve  $\mathcal{N} = 1$  supersymmetry, the  $\mathcal{N} = 4$  sector gives vanishing contribution and only the  $\mathcal{N} = 2$  sectors contribute. Thus, in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric case, one has

$$\Delta^i = \sum_{I=1}^3 \Delta_I^i(T_I, U_I), \quad (2.5)$$

where the threshold corrections  $\Delta_I^i(T_I, U_I)$  come from the three different  $\mathcal{N} = 2$  planes. In this expression,  $T_I, U_I$ ,  $I = 1, 2, 3$ , are the moduli of the three  $\Gamma_{2,2}$ -lattices associated to the six internal dimensions. Notice that in all  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifold models, there are no  $\mathcal{N} = 1$  sectors. The full  $\beta$ -function coefficient in these  $\mathcal{N} = 1$  theories is thus

$$b^i = \sum_{I=1}^3 b_I^i, \quad b_I^i = \frac{1}{2} \sum_{a,b} \mathcal{Q}_b^a \mathcal{P}_i^2 \tau_2 Z_I^a \Big|_{v=\bar{w}=0}, \quad (2.6)$$

where  $Z_I^a$  is the contribution from the plane  $I$ , and the modular covariant helicity operator  $\mathcal{Q}_b^a$  can be replaced by  $\frac{i}{\pi} \partial_\tau \log \theta_b^a$ , since the  $-\frac{i}{\pi} \partial_\tau \log \eta$  contribution is proportional to zero, due to the preservation of supersymmetry.

Our goal is to derive the analogous structure of the threshold corrections to the couplings and to the effective potential in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifold models, where  $\mathcal{N} = 1$  supersymmetry is spontaneously broken “à la Stringy Scherk–Schwarz”. This is done in the context of the moduli-deformed fermionic construction, where the dependence in the moduli  $T_I, U_I$ ,  $I = 1, 2, 3$ , of the three  $\Gamma_{2,2}$ -lattices are implemented. For this purpose, we need the generic form of the associated partition functions, which is found by first following the rules of the fermionic construction and then implementing the moduli deformations, as explained in [Appendix A](#). We obtain in this way not only the generic form of the partition function in symmetric  $\mathbb{Z}_2 \times \mathbb{Z}_2$  orbifolds, but also in non-left/right-symmetric ones.

Limiting ourselves to the continuous deformations parameterized by  $T_I, U_I$ , but including however all possible  $\frac{1}{2}$ -discrete Wilson lines, the generic modular invariant partition function turns out to be

$$\begin{aligned} Z(2v, 2\bar{w}) &= \frac{1}{\tau_2(\eta\bar{\eta})^2} \frac{1}{2} \sum_{a,b} \frac{1}{4} \sum_{H_I, G_I} \frac{1}{2^N} \\ &\times \sum_{h_I^i, \hat{h}_I^i, g_I^i, \hat{g}_I^i} e^{i\pi(a+b+ab)} \frac{\theta_b^a(2v)}{\eta} \frac{\theta_{b+G_2}^{a+H_2}}{\eta} \frac{\theta_{b+G_1}^{a+H_1}}{\eta} \frac{\theta_{b+G_3}^{a+H_3}}{\eta} \\ &\times S \left[ \begin{matrix} a, h_I^i, \hat{h}_I^i, H_I \\ b, g_I^i, \hat{g}_I^i, G_I \end{matrix} \right] Z_{2,2} \left[ \begin{matrix} h_1^i, \hat{h}_1^i \\ g_1^i, \hat{g}_1^i \end{matrix} \middle| \begin{matrix} H_2 \\ G_2 \end{matrix} \right] Z_{2,2} \left[ \begin{matrix} h_2^i, \hat{h}_2^i \\ g_2^i, \hat{g}_2^i \end{matrix} \middle| \begin{matrix} H_1 \\ G_1 \end{matrix} \right] Z_{2,2} \left[ \begin{matrix} h_3^i, \hat{h}_3^i \\ g_3^i, \hat{g}_3^i \end{matrix} \middle| \begin{matrix} H_3 \\ G_3 \end{matrix} \right] \\ &\times Z_{0,16} \left[ \begin{matrix} h_I^i, \hat{h}_I^i, H_I \\ g_I^i, \hat{g}_I^i, G_I \end{matrix} \right] (2\bar{w}), \end{aligned} \quad (2.7)$$

in terms of which the effective potential can be expressed as

$$V_{\text{eff}} = -\frac{1}{(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2^2} Z|_{v=\bar{w}=0}. \quad (2.8)$$

In Eq. (2.7), the variable  $\bar{w}$  refers to a gauge group factor realized by the  $Z_{0,16}$  block but may have been implemented in one of the  $Z_{2,2}$ 's (see the following). Our notations are as follows:

- $(H_1, G_1)$  and  $(H_2, G_2)$  are integer modulo 2, associated to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  action, whose generators twist the internal coordinates  $X^{6,7,8,9}$  and  $X^{4,5,8,9}$ , respectively. We denote  $(H_3, G_3) \equiv -(H_1 + H_2, G_1 + G_2)$ , which is associated to the diagonal action. It is then natural to separate the contributions of the partition function in the following sectors:
  - The  $\mathcal{N} = 4$  sector, which corresponds to  $(H_1, G_1) = (H_2, G_2) = (H_3, G_3) = (0, 0)$ .
  - Three  $\mathcal{N} = 2$  twisted sectors, *i.e.* the so-called complex planes:
    - Complex plane  $I = 1$ :  $(H_1, G_1) = -(H_3, G_3) \neq (0, 0)$  with  $(H_2, G_2) = (0, 0)$ .
    - Complex plane  $I = 2$ :  $(H_2, G_2) = -(H_3, G_3) \neq (0, 0)$  with  $(H_1, G_1) = (0, 0)$ .
    - Complex plane  $I = 3$ :  $(H_1, G_1) = -(H_2, G_2) \neq (0, 0)$  with  $(H_3, G_3) = (0, 0)$ .
  - The  $\mathcal{N} = 1$  twisted sector:  $(H_1, G_1) \neq (0, 0)$ ,  $(H_2, G_2) \neq (0, 0)$ ,  $(H_3, G_3) \neq (0, 0)$ .

As we will explain later in more details,  $\mathcal{N} = 4, 2, 1$  denotes in the above list the number of fermionic zero modes present in each sector, when no spontaneous breaking of supersymmetry to  $\mathcal{N} = 0$  is implemented. Indeed, the (extended) supersymmetry of each sector may or may not be in a spontaneously broken phase,  $\mathcal{N} = 4, 2, 1 \rightarrow \mathcal{N} = 0$ , depending on the choice of  $S$  introduced below.

- $(h_I^i, g_I^i)$ ,  $(\hat{h}_I^i, \hat{g}_I^i)$ ,  $i = 1, 2$ ,  $I = 1, 2, 3$ , are integer modulo 2.  $(h_I^i, g_I^i)$  are shifts and  $(\hat{h}_I^i, \hat{g}_I^i)$  are “dual shifts” of the three untwisted  $\Gamma_{2,2}$ -lattices, which are given as sums over two momenta  $m_I^i$  and two winding numbers  $n_I^i$  associated to each complex plane  $I$  (see Appendix A).
- The contribution of the six internal coordinates (shifted by  $(h_I^i, g_I^i)$ , dual shifted by  $(\hat{h}_I^i, \hat{g}_I^i)$  and twisted by  $(H_I, G_I)$ ), is given in the second line of Eq. (2.7), in terms of the  $(2, 2)$ -conformal blocks  $Z_{2,2} \left[ \begin{smallmatrix} h_I^i, \hat{h}_I^i \\ g_I^i, \hat{g}_I^i \end{smallmatrix} \middle| \begin{smallmatrix} H_I \\ G_I \end{smallmatrix} \right]$ ,  $I = 1, 2, 3$ .
- The fact that the shifts  $(h_I^i, g_I^i)$ , the dual shifts  $(\hat{h}_I^i, \hat{g}_I^i)$  and the twists  $(H_I, G_I)$  are not in general independent leads to an effective normalization factor  $1/2^N$  in the partition function, with  $N$  the number of independent pairs  $(h_I^i, g_I^i)$  and  $(\hat{h}_I^i, \hat{g}_I^i)$ .
- $S$  is a phase that can implement the breaking of  $\mathcal{N} = 1$  spacetime supersymmetry to  $\mathcal{N} = 0$ . When  $S \left[ \begin{smallmatrix} a, h_I^i, \hat{h}_I^i \\ b, g_I^i, \hat{g}_I^i \end{smallmatrix} \middle| \begin{smallmatrix} H_I \\ G_I \end{smallmatrix} \right] \equiv 1$ , the theory is  $\mathcal{N} = 1$  supersymmetric. The latter can be broken spontaneously “à la Stringy Scherk–Schwarz” once some of the 10-dimensional helicity characters (R-parity charges)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a+H_1 \\ b+G_1 \end{pmatrix}, \quad \begin{pmatrix} a+H_2 \\ b+G_2 \end{pmatrix}, \quad \begin{pmatrix} a+H_3 \\ b+G_3 \end{pmatrix} \quad (2.9)$$

are coupled with the lattice charges, *i.e.* with some shifts  $(h_I^i, g_I^i)$  and/or dual shifts  $(\hat{h}_I^i, \hat{g}_I^i)$ .

- Finally, the contribution of the 32 extra right-moving worldsheet fermions is denoted  $Z_{0,16} \left[ \begin{smallmatrix} h_I^i, \hat{h}_I^i \\ g_I^i, \hat{g}_I^i \end{smallmatrix} \middle| \begin{smallmatrix} H_I \\ G_I \end{smallmatrix} \right]$ . In the absence of shifts, dual shifts and twists,  $Z_{0,16}$  is the partition function associated to the  $E_8 \times E_8$  or  $SO(32)$  root lattice. When shifts, dual shifts or twists are non-trivial, the initial gauge group is broken to a product of lower dimensional subgroups

(modulo some stringy extended symmetry points). Therefore, the role of the non-trivial (dual) shifts and twists is to generate non-zero discrete and continuous Wilson lines. According to the fermionic construction rules, the choice of (dual) shifts and twists in realistic models is such that the right-moving gauge group contains an  $SO(10)$  factor, which is further broken to a subgroup that includes the desired standard model gauge group, coupled to acceptable particle content, with three generations (see for instance Ref. [23]).

If no particular attention is devoted to the choice of shifts  $(h_I^i, g_I^i)$  and dual shifts  $(\hat{h}_I^i, \hat{g}_I^i)$ , when supersymmetry is broken to  $\mathcal{N} = 0$  “à la Stringy Scherk–Schwarz”, the resulting  $\mathcal{N} = 0$  model may suffer from the so-called decompactification problem. The reason for this is related to the supersymmetry breaking scale, which is fixed by the inverse of the characteristic size  $R$  of the internal compactified dimensions involved in the breaking,  $m_{\frac{3}{2}} = \mathcal{O}(M_s/R)$ . Indeed, in order to have a small supersymmetry breaking scale compared to the string scale,  $m_{\frac{3}{2}} = 10^{-14} M_s$ ,  $R$  must be enormous. Consequently, when the threshold corrections due to the tower of Kaluza–Klein states are proportional to the volume of the large extra dimensions and dressed with a negative  $\beta$ -function coefficient, the perturbative expansion is invalidated [15,18]. However, this is not always the case. The next section is devoted to the presentation of the simplest example, where such a volume term is not generated.

### 3. The $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ sector

The partition function (2.7) can be separated in sectors according to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  action. In this section, we focus on the  $\mathcal{N} = 4$  sector  $(H_1, G_1) = (H_2, G_2) = (0, 0)$ , which can be spontaneously broken to  $\mathcal{N} = 0$ , when the SSS phase  $S$  is non-trivial. In this case, the induced contribution to the thresholds yields a logarithmic dependence on the volume of the internal directions involved in the susy breaking. Actually, the threshold corrections of the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  sector appearing in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifold models are smaller by a factor 4, compared to those of the full “mother”  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  theory. As a first step, we compute here the threshold corrections in an  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  theory and will remind that in the final result a factor of  $\frac{1}{4}$  arising from a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  projection must be included. We will present in detail the simple case, where a single factorized circle is involved in the process of supersymmetry breaking. This example can be considered as an introduction, since Sections 4–6 will present the analysis valid in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models obtained by moduli-deformed fermionic constructions and where only the  $\mathbb{Z}_2$  action parameterized by  $(H_1, G_1)$  is freely acting.

In an  $\mathcal{N} = 4$  model, two possibilities may arise once a phase  $S$  is introduced. If  $S$  is independent of  $(a, b)$ , then the  $\mathcal{N} = 4$  supersymmetry is unbroken. In this case, the contribution of the worldsheet fermions to the partition function yields

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \theta_b^a(2v) \theta_b^a(v)^3 = \theta_1^1(v)^4 = \mathcal{O}(v^4), \quad (3.1)$$

where we use the Jacobi  $\theta$ -function identity and the relation  $\theta_1^1(v|\tau) = 2\pi\eta^3(\tau)v + \mathcal{O}(v^3)$ . Therefore, the partition function (and effective potential) vanish. Similarly, the helicity insertion, which defines the corrections to the coupling constants, gives

$$\frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \mathcal{Q}_b^a(2v) \theta_b^a(2v) \theta_b^a(v)^3 = \frac{1}{16\pi^2} \partial_v^2 \left( \theta_1^1(v)^4 \right) = \mathcal{O}(v^2), \quad (3.2)$$

which shows that the gauge coupling thresholds vanish as well.



The second possibility is when the phase  $S$  couples non-trivially the helicity charges  $(a, b)$ , with the shifts and/or dual shifts of the internal lattice. This will break spontaneously the  $\mathcal{N} = 4$  supersymmetry to  $\mathcal{N} = 0$ . In order to simplify our discussion in this section, we restrict ourselves to the case where only one  $S^1$  cycle is involved in the susy breaking, and is very large. In this direction, we also consider shifts only,  $(h_1^1, g_1^1)$  we denote as  $(h, g)$ , and take

$$S = e^{i\pi(ah+bg+hg)}. \quad (3.3)$$

Moreover, we specialize to the case where the  $S^1$  shifted lattice is factorized,

$$\Gamma_{6,6+16}[\frac{h}{g}] = \Gamma_{1,1}[\frac{h}{g}](R_1) \Gamma_{5,21}[\frac{h}{g}], \quad (3.4)$$

where  $\Gamma_{5,21}[\frac{h}{g}]$  is a shifted lattice associated to the remaining 5 internal coordinates and the 32 right-moving worldsheet fermions of the heterotic string.<sup>3</sup> For instance, the dependence of the  $\Gamma_{5,21}$ -lattice on  $(h, g)$  may induce a Higgs mechanism by acting on the right-moving worldsheet degrees of freedom. In any case, due to our assumptions, this dependence must not imply a participation of the  $\Gamma_{5,21}$  moduli in the super-Higgs mechanism, which would otherwise induce a very large gravitino mass. The  $S^1$  shifted lattice,  $\Gamma_{1,1}[\frac{h}{g}]$ , admits two representations, Hamiltonian or Lagrangian, which are related to one another by Poisson resummation on the momentum quantum number  $m$  [15,18]:

$$\begin{aligned} \Gamma_{1,1}[\frac{h}{g}](R_1) &= \sum_{m,n} (-)^{mg} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \quad \text{where} \quad p_{\frac{L}{R}} = \frac{1}{\sqrt{2}} \left[ \frac{m}{R_1} \pm \left( n + \frac{h}{2} \right) R_1 \right] \\ &= \frac{R_1}{\sqrt{\tau_2}} \sum_{n,\tilde{m}} e^{-\frac{\pi R_1^2}{\tau_2} \left| (\tilde{m} + \frac{g}{2}) + (n + \frac{h}{2}) \tau \right|^2}. \end{aligned} \quad (3.5)$$

In fact, restricting the internal lattice to the above factorized form will not affect the asymptotic behavior of the threshold corrections for large  $R_1$ .

Because of the non-trivial correlation of the helicity and lattice charges through the SSS susy breaking phase, both the partition function and the coupling constant corrections are not zero. Indeed, in the partition function, the worldsheet fermions and SSS phase give

$$\begin{aligned} \frac{1}{2} \sum_{a,b} (-)^{a+b+ab} e^{i\pi(ag+bh+hg)} \theta[b^a]^4 &= \frac{1}{2} \sum_{A,B} e^{i\pi(A+B+AB+h+g)} \theta \left[ \frac{A+h}{B+g} \right]^4 \\ &= e^{i\pi(h+g+1)} \theta \left[ \frac{1-h}{1-g} \right]^4, \end{aligned} \quad (3.6)$$

which contribute to the effective potential when  $(h, g) \neq (0, 0)$  [24]. Moreover, using the above equation, the integrand involved in the gauge threshold corrections becomes

$$\begin{aligned} \frac{1}{2} \sum_{a,b} \mathcal{Q}[b^a] \left( \mathcal{P}_i^2 - \frac{k^i}{4\pi\tau_2} \right) \tau_2 Z[b^a] \Big|_{v=\bar{w}=0} \\ = \frac{1}{2} \sum_{h,g} e^{i\pi(h+g+1)} \frac{i}{\pi} \left( \frac{1}{4} \partial_\tau \theta \left[ \frac{1-h}{1-g} \right]^4 - (\partial_\tau \log \eta) \theta \left[ \frac{1-h}{1-g} \right]^4 \right) \\ \times \frac{1}{\eta^{12} \bar{\eta}^{24}} \Gamma_{1,1}[\frac{h}{g}](R_1) \left( \mathcal{P}_i^2(2\bar{w}) - \frac{k^i}{4\pi\tau_2} \right) \Gamma_{5,21}[\frac{h}{g}](2\bar{w}) \Big|_{\bar{w}=0}. \end{aligned} \quad (3.7)$$

<sup>3</sup> In  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models,  $\Gamma_{5,21}$  is further factorized as in Eq. (2.7).

The second part of the helicity operator  $\mathcal{Q}_b^{[a]}$  proportional to  $\partial_\tau \log \eta$  gives non-trivial contribution, when supersymmetry is broken to  $\mathcal{N} = 0$  i.e. when  $(h, g) \neq (0, 0)$ .

To perform the integral over the fundamental domain, one can use the unfolding method introduced in Ref. [25] and used in [15,18,19,26]. Defining  $N = 2n + h$  and  $\tilde{M} = 2\tilde{m} + g$ , when  $R_1$  is sufficiently large to guaranty the absolute convergences of the series, one can map the integral over the fundamental domain  $\mathcal{F}$  into an integral over  $\mathcal{F}$  restricted to the pair  $(N, \tilde{M}) = (0, 0)$ , plus an integral over the “upper half strip”  $(-\frac{1}{2} < \tau_2 < \frac{1}{2}, \tau_2 > 0)$  restricted to  $N = 0, \tilde{M} \neq 0$ . In the strip representation, the winding contributions to the fundamental domain integral are mapped to the momentum contributions in the ultraviolet region of the strip,  $\tau_2 < 1$ . In our case, all integrands with  $N = 0$  (i.e.  $n = h = 0$ ) and  $\tilde{M}$  even (i.e.  $g = 0$ ) preserve  $\mathcal{N} = 4$  and therefore vanish, as shown in Eq. (3.2). This is fundamental, since the key point to not have a contribution to the thresholds proportional to a large volume ( $R_1$  in the present case) is that the integrand with  $(N, \tilde{M}) = (0, 0)$  vanishes. Thus, we are left with an integral over the strip, with  $(h, g) = (0, 1)$ ,

$$\Delta^i = \lim_{\mu \rightarrow 0} \left[ \int_{||} \frac{d^2 \tau}{\tau_2} \frac{1}{2} \frac{i}{\pi} \left( \frac{1}{4} \frac{\partial_\tau \theta[l_0^1]^4}{\eta^{12}} - (\partial_\tau \log \eta) \frac{\theta[l_0^1]^4}{\eta^{12}} \right) \frac{R_1}{\sqrt{\tau_2}} \sum_{\tilde{m}} e^{-\frac{\pi R_1^2}{4\tau_2} (2\tilde{m}+1)^2 - \pi \mu^2 \tau_2} \right. \\ \left. \times \left( \mathcal{P}_i^2(2\bar{w}) - \frac{k_i}{4\pi \tau_2} \right) \frac{\Gamma_{5,21}[l_1^0](2\bar{w})}{\tilde{\eta}^{24}} - b^i \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} e^{-\pi \mu^2 \tau_2} \right] \Big|_{\bar{w}=0} + b^i \ln \frac{2e^{1-\gamma}}{\pi \sqrt{27}}. \quad (3.8)$$

In Eq. (3.8), we introduced a small mass  $\mu$  in order to regulate the infrared divergences in the large  $\tau_2$  limit [20]. Other ways to regularize the infrared regime have been proposed recently [26] and have the advantage of preserving in a very elegant way both worldsheet and target space dualities. Our results, however, do not depend of the regularization scheme.

The would be tachyonic level appearing in the right-moving sector is projected out by the level matching condition induced *via*  $\tau_1$ -integration over the strip. In the large  $R_1$  limit, the massive string states give exponentially suppressed contributions to the integral over  $\tau_2$  and can be consistently neglected. The dominant contribution comes from the massless level and even if supersymmetry is broken, there are no-tachyons arising from the left-moving sector. More specifically, we have

$$\left( \frac{i}{\pi} \partial_\tau \log \theta[l_0^1] - \frac{i}{\pi} \partial_\tau \log \eta \right) \frac{\theta[l_0^1]^4}{\eta^{12}} = \left( -\frac{1}{4} + \frac{1}{12} \right) 16 + \mathcal{O}(q) = -\frac{8}{3} + \mathcal{O}(q), \quad (3.9)$$

which is an expected result, since the constant term in the above  $q$ -expansion must be proportional to the  $\beta$ -function contribution of the bosons of the  $\mathcal{N} = 4$  vector multiplets. On the contrary, the gauge group contribution comes from the  $\mathcal{P}_i^2$  charge operator, which acts on the right-moving sector. Actually, in our conventions, the  $\beta$ -function contributions of massless degrees of freedom are:

$$b(\text{gauge boson}) = -\frac{11}{3} C(\mathcal{R}), \quad b(\text{real scalar}) = \frac{1}{6} C(\mathcal{R}), \\ b(\text{Majorana fermion}) = \frac{2}{3} C(\mathcal{R}), \quad (3.10)$$

where  $C(\mathcal{R}) \delta^{ab} = \text{Tr}(T^a T^b)$  is the group factor coefficient associated to the generators  $T^a$  in the representation  $\mathcal{R}$  of  $G^i$ . In an  $\mathcal{N} = 4$  vector multiplet,  $\mathcal{R}$  is the adjoint representation, and there

are 6 real scalars and 4 Majorana gauginos per gauge boson, leading to  $b(\text{bosons}) = -\frac{8}{3} C(\mathcal{R})$  and  $b(\text{fermions}) = \frac{8}{3} C(\mathcal{R})$ . When supersymmetry is unbroken, the  $\mathcal{N} = 4$   $\beta$ -functions vanish. However, in our case, supersymmetry is spontaneously broken *via* the SSS mechanism. The gravitinos and gauginos are getting masses that can be read in the Hamiltonian form of the  $\Gamma_{1,1}[\frac{0}{1}]$ -lattice in Eq. (3.5) and are proportional to the inverse of the internal radius,

$$m_{\frac{3}{2}}^2 = m_{\frac{1}{2}}^2 = \frac{M_s^2}{R_1^2}, \quad (3.11)$$

while the gauge bosons and scalars remain massless,

$$m_1^2 = m_0^2 = 0. \quad (3.12)$$

Thus, the logarithmic behavior of the  $\beta$ -function is fully controlled by the massless bosons, while the main corrections in the thresholds come from the tower of states organized by the shifted  $\Gamma_{1,1}[\frac{0}{1}](R_1)$ -lattice.

Neglecting in Eq. (3.8) the exponentially suppressed contributions for large radius,  $\Delta^i$  gets simplified enormously,

$$\Delta^i = b^i \Delta - k^i Y, \quad (3.13)$$

where  $b^i \Delta$  comes from the  $\mathcal{P}_i^2$  action and  $k^i Y$  is the universal contribution arising from its modular covariant term  $\frac{k^i}{4\pi\tau_2}$ . The former is

$$\begin{aligned} \Delta &= \lim_{\mu \rightarrow 0} \left[ R_1 \sum_{\tilde{m}} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\pi R_1^2}{4\tau_2} (2\tilde{m}+1)^2} e^{-\pi\tau_2\mu^2} - \int_1^{+\infty} \frac{d\tau_2}{\tau_2} e^{-\pi\tau_2\mu^2} \right] - \ln \pi - \gamma + \dots \\ &= \lim_{\mu \rightarrow 0} \left[ 2 \sum_{\tilde{m}} \frac{1}{|2\tilde{m}+1|} e^{-\pi R_1 |2\tilde{m}+1|\mu} - \Gamma(0, \pi\mu^2) \right] - \ln \pi - \gamma + \dots, \end{aligned} \quad (3.14)$$

where the dots stand for  $\mathcal{O}(e^{-cR_1})$  corrections, with  $c$  positive and of the order of the lowest mass  $M_0$  of the massive spectrum divided by  $M_s$ .<sup>4</sup> In the above expression,  $\Gamma(s, x)$  is the upper incomplete  $\Gamma$ -function. Using the fact that  $\Gamma(0, x) = -\ln(x) - \gamma + \mathcal{O}(x)$ , one finally finds

$$\Delta = \lim_{\mu \rightarrow 0} \left[ 2 \ln \left( \frac{1 + e^{-\pi R_1 \mu}}{1 - e^{-\pi R_1 \mu}} \right) + \ln \mu^2 \right] + \dots = -\log \left( \frac{\pi^2}{4} R_1^2 \right) + \dots. \quad (3.15)$$

For the determination of  $Y$ , the infrared regulator  $\mu$  is not needed since the integral is infrared convergent,

$$Y = \frac{C_0}{4\pi} \sum_{\tilde{m}} \int_0^{+\infty} \frac{d\tau_2}{\tau_2^{5/2}} R_1 e^{-\frac{\pi R_1^2}{4\tau_2} (2\tilde{m}+1)^2} + \dots = \frac{7\zeta(3)}{4\pi^2} \frac{C_0}{R_1^2} + \dots. \quad (3.16)$$

In (3.16),  $C_0$  is the product of the contribution of the helicity operator  $\mathcal{Q}_b^{[a]}$  acting on the left-moving sector,  $-\frac{8}{3}$ , with a coefficient  $2 + d_G - n_F$  associated to the right-moving sector,

$$C_0 = \frac{1}{2} \sum_{a,b} \mathcal{Q}_b^{[a]} \tau_2 Z_b^{[a]} \Big|_{q^0 \bar{q}^0, v=0} = -\frac{8}{3} (2 + d_G - n_F). \quad (3.17)$$

<sup>4</sup>  $M_0$  depends on the moduli appearing in the  $\Gamma_{5,21}[\frac{h}{g}]$ -lattice and is at most equal to  $M_s$ .

$d_G$  is the number of vector bosons in the  $\mathcal{N} = 4$  vector multiplets of the parent  $\mathcal{N} = 4$  theory that remain massless after spontaneous breaking to  $\mathcal{N} = 0$ . In other words,  $d_G$  is the dimension of the gauge group. Similarly,  $4n_F$  is the number of Majorana fermions in the  $\mathcal{N} = 4$  vector multiplets of the parent  $\mathcal{N} = 4$  theory that remain massless after spontaneous breaking to  $\mathcal{N} = 0$ . When the shifts  $(h, g)$  are not acting on the right-moving sector, then  $n_F = 0$ . However, in the generic case,  $n_F$  is non-trivial, as is the case in the examples presented in Sections 5.2 and 5.3. Therefore, the corrections to the coupling constants in this  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  model are

$$\Delta^i = b^i \Delta - k^i Y = -b^i \log \left( \frac{\pi^2}{4} R_1^2 \right) + k^i \frac{14\zeta(3)}{3\pi^2} \frac{2 + d_G - n_F}{R_1^2} + \mathcal{O}(e^{-cR_1}). \quad (3.18)$$

The dangerous volume dependence (linear term in  $R_1$ ) is absent, and the reason for this is the restoration of the  $\mathcal{N} = 4$  supersymmetry in the  $R_1 \rightarrow \infty$  limit. Since the universal contribution  $Y$  scales like  $m_{\frac{3}{2}}^2/M_s^2$ , it is a tiny correction to the logarithmic term and may be neglected.

As said at the beginning of this section, the contribution of the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  sector in a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  model is obtained from Eq. (3.18) by changing  $b^i \rightarrow b^i/4$  and  $C_0 \rightarrow C_0/4$ , where the  $\beta$ -function  $b^i$  and  $C_0$  refer to the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  parent theory. However, the presence of  $\mathcal{N} = 2$  sectors requires more attention in the choice of susy breaking (dual) shifts. For instance, an  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  model containing a sector of the form

$$\frac{S^1}{\mathbb{Z}_2^{\text{shift}}} \times \frac{T^4}{\mathbb{Z}_2}, \quad (3.19)$$

where the circle of radius  $R_1$  is shifted as before to break susy spontaneously to  $\mathcal{N} = 0$ , will contain a contribution to the thresholds arising from the integration over  $\mathcal{F}$  of the lattice term with  $(N, \tilde{M}) = (0, 0)$ , which is proportional to the large radius  $R_1$ . This contribution arises from an  $\mathcal{N} = 2$  preserving sector, which therefore *does not vanish* as is the case when  $\mathcal{N} = 4$  is preserved. On the contrary, an  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  model based on an internal space containing a factor

$$\frac{S^1/\mathbb{Z}_2^{\text{shift}} \times T^3}{\mathbb{Z}_2} \quad (3.20)$$

is safe. The reason for this is that the only  $R_1$ -dependent contribution to the partition function arises from the untwisted sector, which realizes an  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  spontaneous breaking. Unfortunately, there is no model based on a single large  $S^1$  shifted direction that realizes a SSS spontaneous breaking of  $\mathcal{N} = 1$  supersymmetry to  $\mathcal{N} = 0$  and solves the decompactification problem. Therefore, we proceed in the next section with the more sophisticated case where two internal shifted directions involved in the breaking are large.

#### 4. $\mathcal{N} = 4$ and 1st plane contributions: $(H_2, G_2) = (0, 0)$

From now on, we come back to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models defined in Eq. (2.7). In this section and the following, we develop a sector by sector analysis of the contributions to the gauge threshold corrections and effective potential. The susy breaking is defined by the SSS phase  $S \left[ \begin{smallmatrix} a, h_I^i, \hat{h}_I^i, H_I \\ b, g_I^i, \hat{g}_I^i, G_I \end{smallmatrix} \right]$  that correlates non-trivially the (dual) shifts and the twists charges with the helicity and R-symmetry charges. However,  $S$  being sector-dependent, it can be trivial ( $S = 1$ ) in some sectors, thus preserving supersymmetry, and non-trivial ( $S \neq 1$ ) in some others, thus inducing a spontaneous breaking of supersymmetry.

In the present section, we focus on the  $\mathcal{N} = 4$  sector  $(H_2, G_2) = (H_1, G_1) = (0, 0)$ , together with the 1st  $\mathcal{N} = 2$  plane  $(H_2, G_2) = (0, 0)$ ,  $(H_1, G_1) \neq (0, 0)$ . We derive here the formal results, and will comment on them physically in Section 5. Both sectors contain sub-sectors, which preserve or break supersymmetry. The contribution of the untwisted internal coordinates  $((H_2, G_2) = (0, 0))$  in the partition function (2.7) involves shifts  $(h_1^i, g_1^i)$ , and we restrict ourselves to the case where no dual shifts are introduced,  $(\hat{h}_1^i, \hat{g}_1^i) \equiv (0, 0)$ . In this class of models, we have

$$Z_{2,2} \left[ \begin{smallmatrix} h_1^i & 0 \\ g_1^i & 0 \end{smallmatrix} \right] = \frac{\Gamma_{2,2} \left[ \begin{smallmatrix} h_1^1 & h_1^2 \\ g_1^1 & g_1^2 \end{smallmatrix} \right]}{(\eta\bar{\eta})^2}, \quad (4.1)$$

which depends on the  $T_1, U_1$  moduli implemented in the moduli-deformed fermionic model, as explained in Appendix A. The shifted lattice dependence on  $T_1, U_1$  (denoted  $T, U$  in this section and Section 5) is

$$\begin{aligned} \Gamma_{2,2} \left[ \begin{smallmatrix} h_1^1 & h_1^2 \\ g_1^1 & g_1^2 \end{smallmatrix} \right] &= \sum_{m^i, n^i} (-)^{m_1 g^1 + m_2 g^2} e^{2i\pi\tau \left[ m_1 \left( n^1 + \frac{h^1}{2} \right) + m_2 \left( n^2 + \frac{h^2}{2} \right) \right]} \\ &\quad \times e^{-\frac{\pi\tau_2}{\text{Im } T \text{ Im } U} \left| T \left( n^1 + \frac{h^1}{2} \right) + T U \left( n^2 + \frac{h^2}{2} \right) - U m^1 + m^2 \right|^2} \\ &= \frac{\sqrt{\det G}}{\tau_2} \sum_{\tilde{m}^i, n^i} e^{-\frac{\pi}{\tau_2} \left[ \tilde{m}^i + \frac{g^i}{2} + \left( n^i + \frac{h^i}{2} \right) \tau \right] (G_{ij} + B_{ij}) \left[ \tilde{m}^j + \frac{g^j}{2} + \left( n^j + \frac{h^j}{2} \right) \bar{\tau} \right]}, \end{aligned} \quad (4.2)$$

where the dictionary between  $T, U$  and the internal metric and antisymmetric tensor in the two associated compact directions is

$$G_{ij} = \frac{\text{Im } T}{\text{Im } U} \begin{pmatrix} 1 & \text{Re } U \\ \text{Re } U & |U|^2 \end{pmatrix}, \quad B_{ij} = \text{Re } T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.3)$$

As explained before, our solution to the decompactification problem requires the breaking of  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  to be spontaneous. This is implemented by imposing the twist action labeled by  $(H_1, G_1) \equiv (H, G)$  to act simultaneously as a shift in the above  $\Gamma_{2,2}$ -lattice. As in the previous section, independent charges  $(h, g)$  that are integer modulo 2 must be used to define the  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  SSS susy breaking phase. In the sectors we consider here, two options parameterized by  $\zeta' = 0$  or 1 can be chosen for the phase  $S$ :

$$\text{In the sectors } (H_2, G_2) = (0, 0), \quad S = e^{i\pi[ag+bh+hg+\zeta'(aG+bH+HG)]}. \quad (4.4)$$

Anticipating the arguments of Sections 5 and 6, when neither of the  $\mathcal{N} = 2$  sectors associated to the 2nd and 3rd planes are realized as a spontaneous breaking of  $\mathcal{N} = 4$  supersymmetry (a fact that we suppose from now on), the moduli  $T_I, U_I$  involved in these planes must not be too far from 1, for the decompactification problem not to occur. In this case,  $(h, g)$  must be associated to the  $\Gamma_{2,2}$ -lattice of the 1st internal 2-torus, for the gravitino masses to be low. Therefore, both shifts  $(h_1^i, g_1^i)$ ,  $i = 1, 2$ , are involved and three classes of two models (labeled by  $\zeta = 0$  or 1) can be analyzed<sup>5</sup>:

<sup>5</sup> The *a priori* remaining cases  $\Gamma_{2,2} \left[ \begin{smallmatrix} h+H & 0 \\ g+G & 0 \end{smallmatrix} \right]$ ,  $\Gamma_{2,2} \left[ \begin{smallmatrix} 0 & h+H \\ 0 & g+G \end{smallmatrix} \right]$  and  $\Gamma_{2,2} \left[ \begin{smallmatrix} h+H & h+H \\ g+G & g+G \end{smallmatrix} \right]$  lead to a volume dependence in the gauge thresholds, arising from the sub-sector  $(h, g) = (H, G) \neq (0, 0)$ , which preserves  $\mathcal{N} = 2$  supersymmetry.

$$\begin{aligned}
a) \quad & \Gamma_{2,2} \left[ \begin{smallmatrix} h+\zeta H, H \\ g+\zeta G, G \end{smallmatrix} \right] \text{ i.e. } (h_1^1, g_1^1) \equiv (h, g) + \zeta(H, G), (h_1^2, g_1^2) \equiv (H, G) \\
b) \quad & \Gamma_{2,2} \left[ \begin{smallmatrix} H, h+\zeta H \\ G, g+\zeta G \end{smallmatrix} \right] \text{ i.e. } (h_1^1, g_1^1) \equiv (H, G), (h_1^2, g_1^2) \equiv (h, g) + \zeta(H, G) \\
c) \quad & \Gamma_{2,2} \left[ \begin{smallmatrix} h+\zeta H, h+(1-\zeta)H \\ g+\zeta G, g+(1-\zeta)G \end{smallmatrix} \right] \text{ i.e. } \\
& (h_1^1, g_1^1) \equiv (h, g) + \zeta(H, G), (h_1^2, g_1^2) \equiv (h, g) + (1-\zeta)(H, G).
\end{aligned} \tag{4.5}$$

In the absence of SSS phase and  $\mathbb{Z}_2^{\text{shift}}$  action parameterized by  $(h, g)$ , the models would describe the partial spontaneous breaking of supersymmetry from  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ , which was considered in [18]. In this reference, it was shown that the pathological volume behaviors of the gauge couplings are absent, thanks to the restoration of  $\mathcal{N} = 4$  supersymmetry in the large volume limit. In the presence of non-trivial SSS phase, the  $\mathbb{Z}_2^{\text{shift}}$  action parameterized by  $(h, g)$  breaks further the supersymmetry to  $\mathcal{N} = 0$ . In this case, the decompactification problem becomes more involved, due to extra contributions coming from the sectors with non-trivial charges  $(h, g)$ .

The separation of the  $(H_2, G_2) = (0, 0)$  sector of the partition function (2.7) in sub-sectors is more transparent once we perform the summation over the helicity charges  $(a, b)$ , keeping the non-trivial characters  $(h, g)$  and  $(H, G)$  fixed<sup>6</sup>:

$$\begin{aligned}
\frac{1}{2} Z \left[ \begin{smallmatrix} h, H \\ g, G \end{smallmatrix} \right] (2v, 2\bar{w}) &= \frac{1}{4\eta^8} \sum_{a,b} e^{i\pi(a+b+ab)} e^{i\pi[ag+bh+hg+\zeta'(aG+bH+HG)]} \\
&\quad \times \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (2v) \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] \theta \left[ \begin{smallmatrix} a+H \\ b+G \end{smallmatrix} \right] \theta \left[ \begin{smallmatrix} a-H \\ b-G \end{smallmatrix} \right] \Gamma_{2,2} \left[ \begin{smallmatrix} h_1^1, h_1^2 \\ g_1^1, g_1^2 \end{smallmatrix} \right] \frac{1}{\bar{\eta}^4} Z_{4,20} \left[ \begin{smallmatrix} h, H \\ g, G \end{smallmatrix} \right] (2\bar{w}) \\
&= \frac{1}{2\eta^8} e^{i\pi[hg+G(1+h+H)]} \theta \left[ \begin{smallmatrix} 1-h \\ 1-g \end{smallmatrix} \right]^2 (v) \theta \left[ \begin{smallmatrix} 1-h+H \\ 1-g+G \end{smallmatrix} \right]^2 (v) \Gamma_{2,2} \left[ \begin{smallmatrix} h_1^1, h_1^2 \\ g_1^1, g_1^2 \end{smallmatrix} \right] \frac{1}{\bar{\eta}^4} \\
&\quad \times Z_{4,20} \left[ \begin{smallmatrix} h, H \\ g, G \end{smallmatrix} \right] (2\bar{w}).
\end{aligned} \tag{4.6}$$

The above result is obtained by redefining  $a = A - h - \zeta' H$ ,  $b = B - g - \zeta' G$  and summing over  $A, B$  equal to 0 or 1. Note that  $\zeta'$  has disappeared, which shows that the two SSS phases  $S$  in Eq. (4.4) are actually equivalent, the different sectors of the theory being simply reshuffled. In Eq. (4.6), the conformal block  $Z_{4,20} \left[ \begin{smallmatrix} h, H \\ g, G \end{smallmatrix} \right]$  for  $(H, G) = (0, 0)$  involves an untwisted lattice  $\Gamma_{4,20} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ , which depends on the moduli  $T_I, U_I$ ,  $I = 2, 3$ . As said before, the latter are close to 1 and therefore must not participate in the super-Higgs mechanism that breaks susy to  $\mathcal{N} = 0$ . Otherwise, a gravitino mass close to  $M_{\text{Planck}}$  would be generated in the sub-sector  $(h, g) \neq (0, 0)$ ,  $(H, G) = (0, 0)$  i.e. far above the acceptable 1–10 TeV region. However, the dependence of the  $\Gamma_{4,20}$ -lattice on  $(h, g)$  may induce a Higgs mechanism arising from an action on the right-moving worldsheet degrees of freedom. Several examples will be given in Section 5.

In Eq. (4.6), the number of odd  $\theta$ -functions  $\theta \left[ \begin{smallmatrix} 1+X \\ 1+Y \end{smallmatrix} \right] (v)$ , with  $(X, Y) = (0, 0)$ , counts the preserved supersymmetries, according to the number of fermionic zero modes in each sub-sector. In the following, we use this number of preserved supersymmetries to classify the sub-sectors and derive the effective potential and gauge couplings corrections in each case.

<sup>6</sup> The factor  $\frac{1}{2}$  in the l.h.s. refers to the  $\mathbb{Z}_2^{\text{shift}}$  projection obtained once the sum over  $h$  and  $g$  is performed. The analogous  $\frac{1}{4}$  factor associated to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  twist (or  $\frac{1}{2}$  for a single  $\mathbb{Z}_2$  twist) will be included later.

#### 4.1. A: The exact $\mathcal{N} = 4$ sector $(h, g) = (0, 0)$ , $(H, G) = (0, 0)$

In this sector we denote A,  $\mathcal{N} = 4$  supersymmetry is unbroken. Therefore, the contributions  $V_{\text{eff}A}$  and  $\Delta_A^i$  to the partition function (or effective potential) and to the gauge couplings vanish. This is due to the fact that the partition function (4.6) is in this case proportional to  $\theta[1]_1^4(v) = \mathcal{O}(v^4)$  and the  $\beta$ -functions are of order  $\mathcal{O}(v^2)$ ,

$$\Delta_A^i = 0, \quad V_{\text{eff}A} = 0. \quad (4.7)$$

The four gravitini in this sector are massless,

$$m_{\frac{3}{2}}^i = 0, \quad i = 1, 2, 3, 4. \quad (4.8)$$

#### 4.2. B: The $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$ sector $(h, g) \neq (0, 0)$ , $(H, G) = (0, 0)$

In this sector we denote B, all arguments of the  $\theta$ -functions in Eq. (4.6) are identical but not equal to  $[1]_1$ . The partition function being proportional to  $\theta[1+h]_{1+g}^4(v)$ , both corrections  $V_{\text{eff}B}$  and  $\Delta_B^i$  to the effective potential and to the  $\beta$ -functions are non-vanishing. The four gravitini have equal non-zero masses, which can be read from the Hamiltonian form of the lattice (4.2) (the first equality),<sup>7,8</sup>

$$m_{\frac{3}{2}}^i \equiv m_B = \frac{|\alpha_B U - \text{sign}(\text{Re } U)\beta_B|}{\sqrt{\text{Im } T \text{Im } U}} M_s = \frac{\sqrt{(\alpha_B \text{Im } U)^2 + (\alpha_B |\text{Re } U| - \beta_B)^2}}{\sqrt{\text{Im } T \text{Im } U}} M_s, \quad (4.9)$$

$$i = 1, 2, 3, 4,$$

where we define

$$(\alpha_B, \beta_B) = \begin{cases} (1, 0) & \text{in case a)} \\ (0, 1) & \text{in case b)} \\ (1, 1) & \text{in case c)}. \end{cases} \quad (4.10)$$

In Section 3, we evaluated the coupling constant correction in case a), when only one radius denoted by  $R_1$  was very large. In this regime, the contribution of the remaining  $\Gamma_{5,21}$ -lattice was trivial. However, there are extra contributions when both compact directions in the 1st plane are large. In the following, utilizing the techniques of Ref. [18], we compute the thresholds in cases a), b) and c) in the regime where the complex moduli  $T$  and  $U$  satisfy  $\text{Im } T \gg 1$ ,  $U$  finite, which guaranties  $m_B \ll M_s$ .

Thanks to the Lagrangian expression of the lattice (4.2) (the second equality), the sector  $h = 1$  is exponentially suppressed. Keeping explicitly the sector  $(h, g) = (0, 1)$ , the threshold corrections in sector B are

$$\begin{aligned} \Delta_B^i &= \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left\{ \frac{1}{\eta^4 \bar{\eta}^4} \frac{i}{4\pi} \partial_{\bar{\tau}} \left( \frac{\theta[1]_0^4}{\eta^4} \right) \frac{1}{2} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] \right. \\ &\quad \times \left( \mathcal{P}_i^2(2\bar{w}) - \frac{k_i}{4\pi \tau_2} \right) Z_{4,20} \left[ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right]; (2\bar{w}) - b_B^i \left. \right\} \Big|_{\bar{w}=0} \\ &\quad + b_B^i \log \frac{2e^{1-\gamma}}{\pi \sqrt{27}} + \dots, \end{aligned} \quad (4.11)$$

<sup>7</sup> We display the masses for  $\text{Re}(U)$  in the range  $(-1, 1]$ .

<sup>8</sup> We define  $\text{sign}(0) = +1$ .

where the coefficient  $b_B^i$  is introduced to cancel the infrared divergence and the dots stand for exponentially small contributions for large  $\text{Im } T$  and finite  $U$ . Similarly, the effective potential based on the partition function (4.6) with  $(H, G) = (0, 0)$  is

$$V_{\text{eff}B} = -\frac{1}{(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \frac{\theta[0]_4^4}{\eta^8 \bar{\eta}^4} \frac{1}{2} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] Z_{4,20} \left[ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right] \Big|_{\tilde{w}=0} + \dots \quad (4.12)$$

In the above two expressions, the dressing with the Lagrangian form of the  $\Gamma_{2,2}$ -lattice implies the non-level matched modes as well as the massive (level-matched) physical states to yield exponentially suppressed contributions. As a result, the universal form of the thresholds in sector  $B$ ,

$$\Delta_B^i = b_B^i \Delta_B - k^i Y_B, \quad (4.13)$$

as well as the effective potential take the simple forms obtained from the massless states and associated Kaluza–Klein modes:

$$\begin{aligned} \Delta_B &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left( \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] - 1 \right) + \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}} + \dots, \\ Y_B &= \frac{C_B}{8\pi} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] + \dots, \\ V_{\text{eff}B} &= -\frac{C_V}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] + \dots, \end{aligned} \quad (4.14)$$

where  $C_B = -\frac{8}{3}(2 + d_{G_B} - n_{F_B})$  and  $C_V = 8(2 + d_{G_B} - n_{F_B})$ . In these coefficients,  $d_{G_B}$  is the number of vector bosons in the  $\mathcal{N} = 4$  vector multiplets of the parent  $\mathcal{N} = 4$  theory that remain massless after spontaneous breaking to  $\mathcal{N} = 0$ , *i.e.* the dimension of the gauge group realized in the sector  $B$ . Similarly,  $4n_F$  is the number of Majorana fermions in the  $\mathcal{N} = 4$  vector multiplets of the parent  $\mathcal{N} = 4$  theory that remain massless after spontaneous breaking to  $\mathcal{N} = 0$ . In other words,  $C_V$  is the index that counts the number of massless bosonic degrees of freedom minus the number of massless fermionic degrees of freedom in the  $\mathcal{N} = 0$  sector  $B$ ,

$$C_V = 8(2 + d_{G_B} - n_{F_B}) \equiv \text{massless Bosons} - \text{massless Fermions in the sector } B. \quad (4.15)$$

A simple way to evaluate  $\Delta_B$  is based on the relation between the shifted lattices  $\Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right]$  and the unshifted one,  $\Gamma_{2,2}(T, U)$ . For the cases *a*), *b*) and *c*), we use respectively

$$\begin{aligned} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right](T, U) &= \sum'_{h,g} \Gamma_{2,2} \left[ \begin{smallmatrix} h & 0 \\ g & 0 \end{smallmatrix} \right](T, U) + \dots = 2\Gamma_{2,2} \left( \frac{T}{2}, 2U \right) - \Gamma_{2,2}(T, U) + \dots, \\ \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right](T, U) &= \sum'_{h,g} \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & h \\ 0 & g \end{smallmatrix} \right](T, U) + \dots = 2\Gamma_{2,2} \left( \frac{T}{2}, \frac{U}{2} \right) - \Gamma_{2,2}(T, U) + \dots, \\ \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right](T, U) &= \sum'_{h,g} \Gamma_{2,2} \left[ \begin{smallmatrix} h & h \\ g & g \end{smallmatrix} \right](T, U) + \dots = 2\Gamma_{2,2} \left( \frac{T}{2}, \frac{1+U}{1-U} \right) - \Gamma_{2,2}(T, U) + \dots, \end{aligned} \quad (4.16)$$

where the primes indicate the sums are over  $(h, g) \neq (0, 0)$ . Using the well know integral [19,27]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left( \Gamma_{2,2}(T, U) - 1 \right) + \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}} = -\log \left( 4\pi^2 |\eta(T)|^4 |\eta(U)|^4 \text{Im } T \text{Im } U \right), \quad (4.17)$$



one obtains

$$\Delta_B = -\log \left( \frac{\pi^2}{4} |\theta_1^0(T)|^4 |\theta_{1-\alpha_B}^{1-\beta_B}(U)|^4 \operatorname{Im} T \operatorname{Im} U \right) + \mathcal{O} \left( e^{-c\sqrt{\operatorname{Im} T}} \right), \quad (4.18)$$

where  $c$  is positive and of the order of the lowest mass of the massive spectrum divided by  $M_s$ . This lowest non-vanishing mass depends on the modulus  $U$ , together with the moduli of the  $\Gamma_{4,20}[\frac{h}{g}]$ -lattice present in the sector  $B$  and introduced below Eq. (4.6). Supposing that the order of magnitude of  $U$  is not too far from 1, a fact that will be justified in Section 5, and given the fact that the  $\Gamma_{4,20}[\frac{h}{g}]$ -lattice moduli are also not too far from 1, we have  $c = \mathcal{O}(1)$ . Moreover, since

$$\log |\theta_1^0(T)|^4 = \mathcal{O} \left( e^{-\pi \operatorname{Im} T} \right), \quad (4.19)$$

this contribution can be omitted in Eq. (4.18). Thus, the  $\operatorname{Im} T$  volume dependence of  $\Delta_B$  is only logarithmic. The key point for this is the following. In the integral (4.17), the contribution  $\tilde{m}^i = n^i = 0$  in the unshifted lattice (4.2) is proportional to  $\sqrt{\det G} = \operatorname{Im} T$ , which is responsible for a  $\frac{\pi}{3} \operatorname{Im} T$  dominant contribution in the result. On the contrary, the shifted lattice in  $\Delta_B$  is expressed in Eq. (4.16) as a difference of two unshifted lattices, where the contribution  $\tilde{m}^i = n^i = 0$  cancels out.

For the second part of the thresholds,  $Y_B$ , and the effective potential, we use the fact that the contributions with non-trivial winding numbers  $n^i$  in the lattice (4.2) are exponentially suppressed,

$$\Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_B & \beta_B \end{smallmatrix} \right] = \frac{\operatorname{Im} T}{\tau_2} \sum_{\tilde{m}_1, \tilde{m}_2} e^{-\frac{\pi}{\tau_2} \frac{\operatorname{Im} T}{\operatorname{Im} U} |\tilde{m}_1 + \frac{\alpha_B}{2} + (\tilde{m}_2 + \frac{\beta_B}{2})U|^2} + \dots \quad (4.20)$$

This expression also justifies that, at our level of approximation, we are free to extend the integration domain from  $\mathcal{F}$  to the full upper half strip. This leads

$$\begin{aligned} Y_B &= -\frac{2 + d_{G_B} - n_{F_B}}{3\pi^3} \frac{1}{\operatorname{Im} T} E_{(\alpha_B, \beta_B)}(U|2) + \mathcal{O} \left( e^{-c\sqrt{\operatorname{Im} T}} \right), \\ V_{\text{eff} B} &= -\frac{2 + d_{G_B} - n_{F_B}}{2\pi^7} \frac{1}{(\operatorname{Im} T)^2} E_{(\alpha_B, \beta_B)}(U|3) + \mathcal{O} \left( e^{-c\sqrt{\operatorname{Im} T}} \right), \end{aligned} \quad (4.21)$$

where we have defined “shifted real analytic Eisenstein series” as

$$E_{(g_1, g_2)}(U|s) = \sum'_{\tilde{m}_1, \tilde{m}_2} \frac{(\operatorname{Im} U)^s}{|\tilde{m}_1 + \frac{g_1}{2} + (\tilde{m}_2 + \frac{g_2}{2})U|^{2s}}. \quad (4.22)$$

In these functions,  $g_1$  and  $g_2$  are integer modulo 2 and the prime means  $\tilde{m}_1 = \tilde{m}_2 = 0$  is excluded from the sum when  $g_1 = g_2 = 0$ . They satisfy modular properties as follows:

$$\begin{aligned} E_{(g_1, g_2)}(M(U)|s) &= E_{(g_1, g_2)M^T}(U|s), \quad \text{where } M(U) = \frac{aU + b}{cU + d}, \\ M &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \end{aligned} \quad (4.23)$$

Note that the sign of the index  $C_V = \text{massless Bosons} - \text{massless Fermions}$  in the sector  $B$  is essential to discuss questions about moduli stabilization [24].

#### 4.3. $C$ : The exact $\mathcal{N} = 2$ sector with $(h, g) = (0, 0)$ , $(H, G) \neq (0, 0)$

The partition function (4.6) associated to this sector, which we will denote by  $C$ , is proportional to  $\theta[1]_1^2(v) \theta[1+G]^{1+H}_1(v) = \mathcal{O}(v^2)$ . Thus, the contribution  $V_{\text{eff}C}$  to the effective potential is zero, while the threshold correction  $\Delta_C^i$  is not vanishing and proportional to an  $\mathcal{N} = 2$   $\beta$ -function coefficient  $b_C^i$ . Two of the four gravitini are massless, while the masses of the other two are given in terms of the  $T$  and  $U$  moduli,

$$m_{\frac{3}{2}}^{1,2} = 0, \quad m_{\frac{3}{2}}^{3,4} \equiv m_C = \frac{\sqrt{(\alpha_C \text{Im } U)^2 + (\alpha_C |\text{Re } U| - \beta_C)^2}}{\sqrt{\text{Im } T \text{Im } U}} M_s, \quad (4.24)$$

where we have

$$(\alpha_C, \beta_C) = \begin{cases} (\zeta, 1) & \text{in case } a) \\ (1, \zeta) & \text{in case } b) \\ (\zeta, 1 - \zeta) & \text{in case } c). \end{cases} \quad (4.25)$$

The threshold corrections in this sector are those of  $\mathcal{N} = 2$  theories that are obtained by an  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  spontaneous susy breaking *via* a free  $\mathbb{Z}_2$  orbifold action. They have been computed in Ref. [18] but we briefly rederive the results we need here.

The Lagrangian form of the lattice (4.2) implies the sector  $H = 1$  to be exponentially suppressed, when  $\text{Im } T \gg 1$  and  $U$  is finite. Keeping explicitly the sector  $(H, G) = (0, 1)$ , one obtains using again  $\theta[1]_1(v|\tau) = 2\pi \eta^3(\tau) v + \mathcal{O}(v^3)$ ,

$$\Delta_C^i = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left\{ \Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_C & \beta_C \end{smallmatrix} \right] \left( \mathcal{P}_i^2(2\bar{w}) - \frac{k_i}{4\pi \tau_2} \right) \bar{\Omega}(2\bar{w}) - b_C^i \right\} \Big|_{\bar{w}=0} + b_C^i \log \frac{2e^{1-\gamma}}{\pi \sqrt{27}} + \dots, \quad (4.26)$$

where  $\bar{\Omega}(2\bar{w}) = \frac{\theta[1]_0^2}{4\eta^2 \bar{\eta}^4} Z_{4,20} \left[ \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right] (2\bar{w})$ .

In fact, since the 4 directions associated to the 2nd and 3rd planes are twisted,  $Z_{4,20} \left[ \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right]$  contains an overall factor  $\eta^2/\theta[1]_0^2$  making  $\bar{\Omega}$  an antiholomorphic function. The contribution  $b_C^i$  to the full  $\beta$ -function coefficient subtracts the infrared divergence. Proceeding as in the sector  $B$ , only the massless contributions dressed by the  $\Gamma_{2,2} \left[ \begin{smallmatrix} 0 & 0 \\ \alpha_C & \beta_C \end{smallmatrix} \right]$ -lattice are non-negligible, leading to formally identical results:

$$\Delta_C^i = b_C^i \Delta_C - k^i Y_C, \quad (4.27)$$

where

$$\Delta_C = -\log \left( \frac{\pi^2}{4} |\theta[1]_1^0(T)|^4 |\theta[1-\beta_C]_{1-\alpha_C}^1(U)|^4 \text{Im } T \text{Im } U \right) + \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right),$$

$$Y_C = -\frac{2+n_{V_C}-n_{H_C}}{3\pi^3} \frac{1}{\text{Im } T} E_{(\alpha_C, \beta_C)}(U|2) + \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right). \quad (4.28)$$

In the above expression,  $n_{V_C}$  and  $n_{H_C}$  are the numbers of massless vector multiplets and hypermultiplets in the sector  $C$ . Thus  $n_{V_C}$  is the dimension of the gauge group  $G_C$  realized in this sector, while

$$\mathcal{I}_C = n_{V_C} - n_{H_C} \quad (4.29)$$

is an index arising naturally from the extended supersymmetry we will denote  $\mathcal{N}_C = 2$ . As in sector  $B$ , the  $|\theta[1]_1^0(T)|^4$ -term can be omitted and the thresholds are only logarithmic in  $\text{Im } T$ . As

said before, it is interesting enough that in this sector the cosmological term vanishes,  $V_{\text{eff}C} = 0$ , thanks to the exact  $\mathcal{N}_C = 2$  supersymmetry.

#### 4.4. $D$ : The exact $\mathcal{N} = 2$ sector with $(h, g) = (H, G)$ , $(G, H) \neq (0, 0)$

We denote this sector as  $D$ . As in sector  $C$ , the partition function (4.6) vanishes, since it is proportional to  $\theta_{[1+G]}^{[1+H]}(v) \theta_{[1]}^2(v) = \mathcal{O}(v^2)$ . There is an exact  $\mathcal{N} = 2$  supersymmetry, which is not that of the sector  $C$ , the two  $\mathcal{N} = 2$  supersymmetries being *not aligned*. The two massless and two massive gravitini are not the same,

$$m_{\frac{3}{2}}^{1,2} \equiv m_D = \frac{\sqrt{(\alpha_D \text{Im } U)^2 + (\alpha_D |\text{Re } U| - \beta_D)^2}}{\sqrt{\text{Im } T \text{Im } U}} M_s, \quad m_{\frac{3}{2}}^{3,4} = 0 \quad (4.30)$$

and the non-vanishing masses are even different to those in sector  $C$ . This is due to the fact that the pairs  $(\alpha_D, \beta_D)$  and  $(\alpha_C, \beta_C)$  are not equal,

$$(\alpha_D, \beta_D) = \begin{cases} (1 - \zeta, 1) & \text{in case } a) \\ (1, 1 - \zeta) & \text{in case } b) \\ (1 - \zeta, \zeta) & \text{in case } c). \end{cases} \quad (4.31)$$

Actually, we see that the sectors  $C$  and  $D$  are replaced by one another under the change  $\zeta \rightarrow 1 - \zeta$ ,

$$\text{sector } C \leftrightarrow \text{sector } D \iff \zeta \rightarrow 1 - \zeta. \quad (4.32)$$

As a result, the threshold corrections to the gauge couplings are

$$\Delta_D^i = b_D^i \Delta_D - k^i Y_D, \quad (4.33)$$

where

$$\begin{aligned} \Delta_D &= -\log \left( \frac{\pi^2}{4} |\theta_{[1]}^0(T)|^4 |\theta_{[1-\alpha_D]}^{[1-\beta_D]}(U)|^4 \text{Im } T \text{Im } U \right) + \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right), \\ Y_D &= -\frac{2 + n_{V_D} - n_{H_D}}{3\pi^3} \frac{1}{\text{Im } T} E_{(\alpha_D, \beta_D)}(U|2) + \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right). \end{aligned} \quad (4.34)$$

$n_{V_D}$  and  $n_{H_D}$  count the massless vector multiplets and hypermultiplets in the sector  $D$ , while

$$\mathcal{I}_C = n_{V_D} - n_{H_D} \quad (4.35)$$

is the index arising from the second non-aligned extended supersymmetry we will denote  $\mathcal{N}_D = 2$ . Of course,  $n_{V_D}$  is nothing but the dimension of the gauge group  $G_D$  realized in this sector. As before, the  $|\theta_{[1]}^0(T)|^4$ -term in  $\Delta_D$  can be omitted and the contribution to the cosmological term vanishes:  $V_{\text{eff}D} = 0$ .

#### 4.5. $E \& F$ : The $\mathcal{N}_{C,D} = 2 \rightarrow \mathcal{N}_{C,D} = 0$ sectors $hG - gH \neq 0$

The previous sectors  $A, B, C, D$  have  $(H, G)$  or  $(h, g)$  equal to  $(0, 0)$ , or  $(H, G) = (h, g)$ . All these conditions are equivalent to saying that the determinant  $\begin{vmatrix} h & H \\ g & G \end{vmatrix}$  vanishes. In the remaining sectors, namely  $E$  and  $F$ , one has  $\begin{vmatrix} h & H \\ g & G \end{vmatrix} \neq 0$ , which implies not only that  $(H, G) \neq (0, 0)$ , but also that  $(h, g) \neq (0, 0)$  and  $(h, g) \neq (H, G)$ . In other words, the supersymmetries  $\mathcal{N}_C = 2$  of sector  $C$  and  $\mathcal{N}_D = 2$  of sector  $D$  are both broken to  $\mathcal{N}_C = 0$  and  $\mathcal{N}_D = 0$ . Indeed, one finds that in the partition function (4.6), the left-moving part (including the four twisted left-moving internal

coordinates) is not vanishing and universal, modulo the dressing with  $\Gamma_{2,2}$  shifted lattices. We display below the refined partition function in case *a*), for the  $\mathcal{N}_C = 0$  sector *E*,

$$\begin{aligned} \frac{1}{2} \left( Z_{[0,1]}^{[1,0]} + Z_{[1,0]}^{[1,1]} + Z_{[1,1]}^{[0,1]} \right) \Big|_{2v,2\bar{w}} &= -8 \frac{\theta_4^2(v) \theta_3^2(v)}{\eta^6 \theta_2^2} \Gamma_{2,2} \left[ \begin{smallmatrix} 1, 0 \\ \zeta, 1 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 1|0 \\ 0|1 \end{smallmatrix} \right] (2\bar{w}) \\ &\quad - 8 \frac{\theta_3^2(v) \theta_2^2(v)}{\eta^6 \theta_4^2} \Gamma_{2,2} \left[ \begin{smallmatrix} 1-\zeta, 1 \\ 1, 0 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 1|1 \\ 1|0 \end{smallmatrix} \right] (2\bar{w}) \\ &\quad + 8 \frac{\theta_2^2(v) \theta_4^2(v)}{\eta^6 \theta_3^2} \Gamma_{2,2} \left[ \begin{smallmatrix} \zeta, 1 \\ 1-\zeta, 1 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 0|1 \\ 1|1 \end{smallmatrix} \right] (2\bar{w}), \end{aligned} \quad (4.36)$$

and for the  $\mathcal{N}_D = 0$  sector *F*,

$$\begin{aligned} \frac{1}{2} \left( Z_{[1,1]}^{[1,0]} + Z_{[1,0]}^{[0,1]} + Z_{[0,1]}^{[1,1]} \right) \Big|_{2v,2\bar{w}} &= +8 \frac{\theta_4^2(v) \theta_3^2(v)}{\eta^6 \theta_2^2} \Gamma_{2,2} \left[ \begin{smallmatrix} 1, 0 \\ 1-\zeta, 1 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 1|0 \\ 1|1 \end{smallmatrix} \right] (2\bar{w}) \\ &\quad + 8 \frac{\theta_3^2(v) \theta_2^2(v)}{\eta^6 \theta_4^2} \Gamma_{2,2} \left[ \begin{smallmatrix} \zeta, 1 \\ 1, 0 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 0|1 \\ 1|0 \end{smallmatrix} \right] (2\bar{w}) \\ &\quad - 8 \frac{\theta_2^2(v) \theta_4^2(v)}{\eta^6 \theta_3^2} \Gamma_{2,2} \left[ \begin{smallmatrix} 1-\zeta, 1 \\ \zeta, 1 \end{smallmatrix} \right] \bar{Z} \left[ \begin{smallmatrix} 1|1 \\ 0|1 \end{smallmatrix} \right] (2\bar{w}). \end{aligned} \quad (4.37)$$

In these expressions, the  $\bar{Z}$ -factors are purely antiholomorphic. The partition functions in case *b*) are obtained from the above ones by exchanging the columns of the  $\Gamma_{2,2}$ -lattices. In case *c*), the first columns of the  $\Gamma_{2,2}$ -lattices are as above, while the second columns are obtained by changing  $\zeta \rightarrow 1 - \zeta$  in the first ones.

The key point here is that once  $\left| \frac{h}{g} \frac{H}{G} \right| \neq 0$ , it is forbidden to have  $h = H = 0$  in the sectors *E* and *F*. Therefore, all individual terms in the associated partition functions are coupled with exponentially suppressed shifted lattices (see Eq. (4.5)), when  $\text{Im } T$  is large and  $U$  finite. This shows explicitly that in sectors *E* and *F*, the contributions to the cosmological term and coupling constants can be neglected,

$$\Delta_{E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right), \quad Y_{E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right), \quad V_{\text{eff},E,F} = \mathcal{O} \left( e^{-c \sqrt{\text{Im } T}} \right). \quad (4.38)$$

## 5. Analysis of the $\mathcal{N} = 4$ and 1st plane contributions

Before investigating the 2nd and 3rd planes contributions in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models where only the 1st  $\mathbb{Z}_2$  action is freely acting, we would like to comment further on the structure of the corrections coming from the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  susy breaking associated to the sectors *A* to *F* in these models. Some explicit examples will also be given. Let us start by collecting the results found in the previous section:

- In sector *A*, the contributions to the effective potential,  $V_{\text{eff},A}$ , and to the gauge thresholds,  $\Delta_A^i$ , are always zero due to the “mother”  $\mathcal{N} = 4$  theory.
- There are two non-aligned  $\mathcal{N}_C = 2$  and  $\mathcal{N}_D = 2$  “daughter” supersymmetries in the sectors *C* and *D*. In the former, the first two gravitini are massless, while in the latter the third and fourth gravitini are massless. Gauge coupling corrections  $\Delta_{C,D}^i$  occur, while there are no contributions to the effective potential,  $V_{\text{eff},C,D} = 0$ .

- The sectors  $E$  and  $F$  are not supersymmetric and correspond to the breaking  $\mathcal{N}_{C,D} = 2 \rightarrow \mathcal{N}_{C,D} = 0$ . However, their contributions  $V_{\text{eff},E,F}$  and  $\Delta_{E,F}^i$  are exponentially suppressed, when  $\text{Im } T$  is large and  $U$  finite.
- The contributions  $V_{\text{eff},B}$  and  $\Delta_B^i$  of the sector  $B$  are the only ones arising from a non-supersymmetric sector. The latter realizes a spontaneous breaking of  $\mathcal{N} = 4$  to  $\mathcal{N} = 0$ . Moreover, the sector  $B$  is the only one that gives a non-vanishing (or non-negligible) cosmological term, which is proportional to  $m_{\frac{3}{2}}^4 \equiv m_B^4 \propto 1/(\text{Im } T)^2$ .
- The non-trivial contributions to the gauge thresholds arise from the sectors  $B$ ,  $C$  and  $D$ . For any model  $a$ ),  $b$ ) or  $c$ ), with  $\zeta = 0$  or  $1$ ,  $(\alpha_B, \beta_B)$ ,  $(\alpha_C, \beta_C)$  and  $(\alpha_D, \beta_D)$  take distinct values among the set  $\{(1, 0), (0, 1), (1, 1)\}$ . In fact, the 6 models realize the  $3!$  allowed permutations of these parameters.

The contributions of the sectors  $A$  to  $F$  are what is required to write the corrections to the gauge coupling constants and to the cosmological term in the  $\mathbb{Z}_2$  non-symmetric orbifold models, where shifts along the untwisted plane realize an  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  spontaneous susy breaking à la SSS. The running gauge couplings can be expressed in terms of the redefined infrared regulator  $Q^2 = \mu^2 \frac{\pi^2}{4}$ , and are valid for  $Q < m_B, m_C, m_D < cM_s$ , where  $c$  is defined below Eq. (4.18). They take the form

$$\begin{aligned} \frac{16\pi^2}{g_i^2(Q)} = & k^i \frac{16\pi^2}{g_s^2} - \frac{1}{2}(b_B^i + b_C^i + b_D^i) \log \frac{Q^2}{M_s^2} \\ & - \frac{1}{2} b_B^i \log \left( |\theta[1-\beta_B](U)|^4 \text{Im } T \text{Im } U \right) \\ & - \frac{1}{2} b_C^i \log \left( |\theta[1-\beta_C](U)|^4 \text{Im } T \text{Im } U \right) \\ & - \frac{1}{2} b_D^i \log \left( |\theta[1-\beta_D](U)|^4 \text{Im } T \text{Im } U \right) + \mathcal{O}\left(\frac{1}{\text{Im } T}\right), \end{aligned} \quad (5.1)$$

while the effective potential is

$$\begin{aligned} V_{\text{eff}} = & \frac{1}{2} V_{\text{eff},B} + \mathcal{O}\left(e^{-c\sqrt{\text{Im } T}}\right) \\ = & -\frac{1}{2} \frac{2 + d_{G_B} - n_{F_B}}{2\pi^7} \frac{1}{(\text{Im } T)^2} E_{(\alpha_B, \beta_B)}(U, 3) + \mathcal{O}\left(e^{-c\sqrt{\text{Im } T}}\right). \end{aligned} \quad (5.2)$$

The factors  $\frac{1}{2}$  in front of the  $\beta$ -function coefficients and in the expression of the potential come from the normalization arising from the  $\mathbb{Z}_2$  orbifold projection. The gravitino mass  $m_{\frac{3}{2}}$  of the  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  model being equal to that of sector  $B$ ,

$$m_{\frac{3}{2}} \equiv m_B = \frac{\sqrt{(\alpha_B \text{Im } U)^2 + (\alpha_B |\text{Re } U| - \beta_B)^2}}{\sqrt{\text{Im } T \text{Im } U}} M_s, \quad (5.3)$$

the cosmological term is proportional to  $m_{\frac{3}{2}}^4$ . Note that no correction of order  $M_s^2 m_{\frac{3}{2}}^2$  occurs. In order to make the physical interpretation of the gauge coupling threshold corrections more transparent, it is convenient to introduce moduli-dependent mass scales,

$$\frac{1}{M_B^2} = \frac{1}{M_s^2} |\theta[1-\beta_B](U)|^4 \text{Im } T \text{Im } U,$$

$$\begin{aligned}\frac{1}{M_C^2} &= \frac{1}{M_s^2} |\theta[{}_{1-\alpha_C}^{1-\beta_C}](U)|^4 \operatorname{Im} T \operatorname{Im} U, \\ \frac{1}{M_D^2} &= \frac{1}{M_s^2} |\theta[{}_{1-\alpha_D}^{1-\beta_D}](U)|^4 \operatorname{Im} T \operatorname{Im} U,\end{aligned}\quad (5.4)$$

in terms of which the coupling constant corrections for  $Q < M_B, M_C, M_D$  take the form

$$\frac{16\pi^2}{g_i^2(Q)} = k^i \frac{16\pi^2}{g_s^2} - \frac{1}{2} b_B^i \log \frac{Q^2}{M_B^2} - \frac{1}{2} b_C^i \log \frac{Q^2}{M_C^2} - \frac{1}{2} b_D^i \log \frac{Q^2}{M_D^2} + \mathcal{O}\left(\frac{1}{\operatorname{Im} T}\right). \quad (5.5)$$

As we are going to see, the behavior of these thresholds depends crucially on the complex structure  $U$ . In particular, the hierarchy between the moduli-dependent scales  $M_B, M_C, M_D$  depends only on  $U$ . To further investigate the qualitative features of the  $U$ -dependence, we can focus on the particular susy breaking pattern of model *a*), with  $\zeta = 0$ , keeping in mind that the gauge coupling thresholds in all six cases *a*), *b*), *c*), with  $\zeta = 0, 1$ , are obtained by permutation of the defining expressions of the threshold scales  $M_{B,C,D}$ . In this case, the shifted lattice involved in the threshold corrections is  $\Gamma_{2,2}[\begin{smallmatrix} 0 & 0 \\ g & G \end{smallmatrix}]$  and the susy breaking scales in sectors  $B, C, D$  are

$$\begin{aligned}m_B &= \frac{|U|}{\sqrt{\operatorname{Im} T \operatorname{Im} U}} M_s, & m_C &= \frac{1}{\sqrt{\operatorname{Im} T \operatorname{Im} U}} M_s, \\ m_D &= \frac{\sqrt{(\operatorname{Im} U)^2 + (1 - |\operatorname{Re} U|)^2}}{\sqrt{\operatorname{Im} T \operatorname{Im} U}} M_s.\end{aligned}\quad (5.6)$$

This shows that the scale at which  $\mathcal{N} = 4$  is spontaneously broken to  $\mathcal{N} = 2$  is  $m_C$ , since  $(g, G) = (0, 1)$  is the value taken by  $(\alpha_C, \beta_C)$ . Similarly, the scale at which supersymmetry is spontaneously broken to  $\mathcal{N} = 0$  is  $m_B$ , since  $(g, G) = (1, 0)$  is the value taken by  $(\alpha_B, \beta_B)$ . These two scales are relatively small compared to  $M_s$ , as is also the third one,  $m_D$ , which emerges for  $(g, G) = (1, 1) = (\alpha_D, \beta_D)$ .

To proceed, we specialize further to the situation where  $\operatorname{Re} U = 0$  and define

$$t = \operatorname{Im} T = R_1 R_2, \quad u = \operatorname{Im} U = \frac{R_2}{R_1}, \quad (5.7)$$

where  $R_1$  and  $R_2$  are the radii of the shifted squared untwisted internal 2-torus. The susy breaking scales become

$$m_B^2 = \frac{u}{t} M_s^2, \quad m_C^2 = \frac{1}{tu} M_s^2, \quad m_D^2 = m_B^2 + m_C^2, \quad (5.8)$$

which implies  $m_D$  is the largest one. The moduli-dependent scales  $M_B$  and  $M_C$  become

$$\begin{aligned}\frac{1}{M_B^2} &= \frac{1}{M_s^2} |\theta_2(iu)|^4 tu = \frac{1}{m_B^2} |\theta_4(i/u)|^4, \\ \frac{1}{M_C^2} &= \frac{1}{M_s^2} |\theta_4(iu)|^4 tu = \frac{1}{m_C^2} |\theta_4(iu)|^4.\end{aligned}\quad (5.9)$$

Utilizing the identity  $|\theta_3(iu)|^4 = |\theta_2(iu)|^4 + |\theta_4(iu)|^4$ , which is valid for pure imaginary arguments, we obtain the moduli-dependent threshold scale related to the  $\mathcal{N}_D = 2$  supersymmetric sector  $D$  as a function of  $M_B$  and  $M_C$ ,

$$\frac{1}{M_D^2} = \frac{1}{M_s^2} |\theta_3(iu)|^4 tu = \frac{1}{M_B^2} + \frac{1}{M_C^2}. \quad (5.10)$$

This shows that in the present case,  $M_D$  is the lowest threshold scale. This example is illuminating. It shows that the scales at which supersymmetry is restored in the sectors  $B, C, D$  are not the associated gravitini masses  $m_{B,C,D}$ . Instead, the relevant scales for supersymmetry restoration are the full threshold scales  $M_{B,C,D}$ , whose hierarchy differs from that of the scales  $m_{B,C,D}$ . For instance, since

$$M_B^2 \sim m_B^2, \quad M_C^2 \sim \frac{m_B^2}{16} e^{\pi/u}, \quad M_D^2 \sim m_B^2, \quad \text{when } u \ll 1, \quad (5.11)$$

the full hierarchy of the threshold scales for small enough  $u$  is  $Q < M_D \leq M_B \leq M_C$ , while we have  $m_B < m_C \leq m_D$ . Moreover, in the limit where  $u$  is very small, the scale  $M_C$  grows exponentially, which gives large corrections to the gauge couplings in Eq. (5.5), proportional to  $1/u = R_1/R_2$ . On the contrary, since

$$M_B^2 \sim \frac{m_C^2}{16} e^{\pi u}, \quad M_C^2 \sim m_C^2, \quad M_D^2 \sim m_C^2, \quad \text{when } u \gg 1, \quad (5.12)$$

the hierarchy of the threshold scales for large enough  $u$  is  $Q < M_D \leq M_C \leq M_B$ , while  $m_C < m_B \leq m_D$ . Furthermore, when  $u$  is very large, the scale  $M_B$ , which grows exponentially with  $u$ , gives rise to large corrections to the couplings in Eq. (5.5), proportional to  $u = R_2/R_1$ . In the end, in both extreme limits summarized by the condition  $u + 1/u \gg 1$ , large linear corrections can destroy the string perturbative expansion, when dressing  $\beta$ -function coefficients are negative. In such cases, one must assume that  $u$  is not too small or large.

In our low energy description, the range of permitted ratios  $u = R_2/R_1$  can be derived by the requirement that the higher threshold scale must be smaller than the scale of the massive states we neglected *i.e.*  $cM_s$ . In general, the lowest threshold scale among  $M_B$ ,  $M_C$  and  $M_D$  in Eq. (5.4) is the one that contains  $\theta_3(U)$  in its definition. As we have just shown, this scale has a simple relation with the highest threshold scale in the extreme limits  $u \gg 1$  or  $u \ll 1$ . The validity constraint in these two limits becomes

$$\frac{1}{16} e^{\pi(u+1/u)} = \frac{M_{\text{high}}^2}{M_{\text{low}}^2} < c^2 \frac{M_s^2}{M_{\text{low}}^2} = c^2 \frac{M_{\text{Planck}}^2}{M_{\text{low}}^{(E)2}}, \quad (5.13)$$

where  $M_{\text{low}}^{(E)}$  is the lowest scale measured in the Einstein frame. Notice that the ratio  $M_{\text{high}}/M_{\text{low}}$  is independent of the frame. This gives the condition:

$$u + \frac{1}{u} < \frac{2}{\pi} \log \left( 4c \frac{M_{\text{Planck}}}{M_{\text{low}}^{(E)}} \right). \quad (5.14)$$

Assuming the lowest supersymmetry breaking scale measured in Einstein frame to be in the 1–10 TeV region, we take  $M_{\text{low}}^{(E)} = \mathcal{O}(10^4)$  GeV, and given the gravity scale  $M_{\text{Planck}} = 2.4 \cdot 10^{18}$  GeV, one finds for  $c = \mathcal{O}(1)$  the permitted values of  $u$ :

$$u + \frac{1}{u} < 22. \quad (5.15)$$

Once  $u$  is in this region, we can write the following interpolating expression for the running gauge couplings, in terms of the physical energy scale measured in string frame,  $Q < cM_s$  (or  $Q^{(E)} \equiv Q/g_s < cM_{\text{Planck}}$  in the Einstein frame). It is valid for all supersymmetry breaking patterns *i.e.* models *a*), *b*) or *c*), with  $\zeta = 0$  or 1, and independently of the  $U$ -dependent hierarchy among the threshold scales  $M_B$ ,  $M_C$  and  $M_D$ :

$$\begin{aligned} \frac{16\pi^2}{g_i^2(Q)} = & k^i \frac{16\pi^2}{g_s^2} - \frac{1}{2} b_B^i \log \left( \frac{Q^2}{Q^2 + M_B^2} \right) - \frac{1}{2} b_C^i \log \left( \frac{Q^2}{Q^2 + M_C^2} \right) \\ & - \frac{1}{2} b_D^i \log \left( \frac{Q^2}{Q^2 + M_D^2} \right). \end{aligned} \quad (5.16)$$

The above expression implements the successive decouplings of the effective threshold mass scales  $M_{B,C,D}$ , which occur when the infrared cut-off scale  $Q$  crosses them.  $Q$  plays the role of a scattering energy scale. For  $Q$  smaller than the three threshold scales, it can be neglected compared to them and one recovers the threshold formula for small  $Q$ , Eq. (5.5). Once  $Q$  becomes larger than one of the threshold scales, the latter can be neglected compared to  $Q$ , which is consistent with the fact that the whole tower of associated thresholds give negligible contribution. In particular:

- In the cases where the susy breaking pattern and the complex structure  $U$  imply  $M_B$  to be the lowest threshold scale, when the physical scale satisfies  $M_B < Q < M_C, M_D$ , the two non-aligned  $\mathcal{N}_C = 2$  and  $\mathcal{N}_D = 2$  supersymmetries are restored. The full  $\mathcal{N} = 4$  supersymmetry is recovered when  $Q$  is above  $M_C$  and  $M_D$ .
- In the cases where the susy breaking pattern and the complex structure  $U$  imply  $M_B$  to be the highest threshold scale, then the model describes a total  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  spontaneous susy breaking, when the physical scale satisfies  $M_C, M_D < Q < M_B$ . When  $Q > M_B$ , the full  $\mathcal{N} = 4$  supersymmetry is restored.

### 5.1. Example 1: Gauge group factor $E_8$ with $n_F = 0$

Before analyzing the contributions of the 2nd and 3rd planes in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models we consider, we would like to present typical examples in the  $\mathbb{Z}_2$  case *i.e.* where  $\mathcal{N} = 4$  supersymmetry is spontaneously broken to  $\mathcal{N} = 2$  and further broken to  $\mathcal{N} = 0$  using the shifts and  $T, U$  moduli of the untwisted  $\Gamma_{2,2}$ -lattice. In fact, the  $\beta$ -function coefficients we are going to focus on can either be deduced by computing those associated to the sectors  $B, C$  and  $D$ , or directly by considering the massless spectrum of the  $\mathcal{N} = 0$  theory.

In our first example, we consider the models whose gauge groups contain a factor  $G^i = E_8$ . The associated affine character in the adjoint representation,  $\bar{E}_8(\bar{\tau})$ , is realized by 16 right-moving Majorana–Weyl worldsheet fermions,

$$\bar{E}_8(\bar{\tau}) = \frac{1}{2} \sum_{\gamma, \delta} \frac{\bar{\theta}[\gamma]_8}{\bar{\eta}^8}. \quad (5.17)$$

The latter is factorized in the right-moving part of the partition function, whose relevant conformal block takes the form

$$Z_{4,20} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] = Z_{4,12} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] \bar{E}_8. \quad (5.18)$$

The adjoint character  $\bar{E}_8$  can be written in terms of those associated to the adjoint and spinorial representations of  $SO(16)$ ,

$$\bar{E}_8 = \bar{O}_{16} + \bar{S}_{16}, \quad (5.19)$$

where our conventions for the holomorphic  $SO(2N)$  characters are



$$\begin{aligned}
O_{2N} &= \frac{\theta[0]_0^N + \theta[1]_1^N}{2\eta^N}, & V_{2N} &= \frac{\theta[0]_0^N - \theta[1]_1^N}{2\eta^N}, \\
S_{2N} &= \frac{\theta[0]_0^N + (-i)^N \theta[1]_1^N}{2\eta^N}, & C_{2N} &= \frac{\theta[0]_0^N - (-i)^N \theta[1]_1^N}{2\eta^N}.
\end{aligned} \quad (5.20)$$

Since the character  $\bar{E}_8$  is factorized, the gauge groups realized in the sectors  $B, C, D$  contain a common factor,  $G_B^i = G_C^i = G_D^i = E_8$ . In sector  $B$ , the  $\beta$ -function coefficient arises from the bosonic part of an  $\mathcal{N} = 4$  vector multiplet (1 gauge boson + 6 real scalars) in the adjoint of  $G_B^i$ . In the sectors  $C$  and  $D$ , the  $\beta$ -function coefficients correspond to  $\mathcal{N}_C = 2$  and  $\mathcal{N}_D = 2$  vector multiplets in the adjoint of  $G_C^i$  and  $G_D^i$ . Thus, we have

$$b_B^i = -\frac{8}{3} C(E_8), \quad b_C^i = -2 C(E_8), \quad b_D^i = -2 C(E_8), \quad (5.21)$$

where  $C(E_8) = 14 + 16 = 30$ . The contribution 14 in  $C(E_8)$  comes from the adjoint of  $SO(16)$ ,  $C(O_{16}) = 14$ , while the contribution 16 comes from the spinorial of  $SO(16)$ ,  $C(S_{16}) = 16$ . Thus, the sector by sector analysis leads to a  $\beta$ -function coefficient in the  $\mathcal{N} = 0$  theory given by

$$b^i = \frac{1}{2}(b_B^i + b_C^i + b_D^i) = -\frac{10}{3} C(E_8) = -100, \quad (5.22)$$

which shows that the gauge theory is asymptotically free.

To cross check this value, we can directly compute  $b^i$  from the point of view of an  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  spontaneously broken theory. The massless spectrum contains the bosonic part of an  $\mathcal{N} = 2$  vector multiplet in the adjoint representation of  $G^i$ , namely 1 gauge boson for 2 real scalars, while the gauginos have become massive:

$$b^i = \left(-\frac{11}{3} + \frac{2}{6}\right) C(E_8) = -\frac{10}{3} C(E_8). \quad (5.23)$$

## 5.2. Example 2: Gauge group factor $SO(16)$ with $n_F \neq 0$

The second models we would like to present have a gauge group factor  $G^i = SO(16)$ . The latter is obtained by coupling non-trivially the lattice shift  $(h, g)$ , with the  $SO(16)$  spinorial representation initially present in the character  $\bar{E}_8$ . The coupling is implemented by a phase as follows:

$$Z_{4,20}[g|_G^h] = Z_{4,12}[g|_G^h] Z_{0,8}[g] \quad \text{where} \quad Z_{0,8}[g] = \frac{1}{2} \sum_{\gamma, \delta} \frac{\bar{\theta}[\gamma]_\delta^8}{\bar{\eta}^8} e^{i\pi(g\gamma + h\delta + hg)}, \quad (5.24)$$

which breaks simultaneously  $E_8 \rightarrow SO(16)$  and supersymmetry to  $\mathcal{N} = 0$ .

The SSS phase changes effectively to

$$S = e^{i\pi[g(a+\gamma) + h(b+\delta)]}. \quad (5.25)$$

This shows clearly that in the sector  $B$ , the fermions of the initially massless  $\mathcal{N} = 4$  vector multiplets in the  $\bar{O}_{16}$  representation (*i.e.* for  $\gamma = 0$ ) become massive, while the bosons remain massless. However, compared to Example 1, the new thing is that the situation is reversed for the states in the  $\bar{S}_{16}$  representation (*i.e.* for  $\gamma = 1$ ): The bosons of the originally massless  $\mathcal{N} = 4$  vector multiplets become massive, while the fermions remain massless. In total, the gauge group factor in the non-supersymmetric sector  $B$  is  $G_B^i = SO(16)$  and the  $\beta$ -function coefficient is

$$b_B^i = -\frac{8}{3} \{C(O_{16}) - C(S_{16})\}. \quad (5.26)$$

Notice that since  $(h, g) = (0, 0)$  in sector  $C$ , the gauge group factor  $G_C^i = E_8$  is unbroken and the associated  $\mathcal{N}_C = 2$  supersymmetric  $\beta$ -function coefficient is identical to that of Example 1,

$$b_C^i = -2 C(E_8) \equiv -2 \{C(O_{16}) + C(S_{16})\}. \quad (5.27)$$

However, in sector  $D$ , where  $(h, g) \neq (0, 0)$ , the  $E_8$  gauge group is broken to  $G_D^i = SO(16)$ , with massless hypermultiplets in the spinorial representation  $\bar{S}_{16}$ . The  $\mathcal{N}_D = 2$  supersymmetric  $\beta$ -function coefficient is thus

$$b_D^i = -2 \{C(O_{16}) - C(S_{16})\}. \quad (5.28)$$

Taking into account the above sector by sector contributions, the  $\beta$ -function coefficient of the  $G^i = SO(16)$  non-supersymmetric gauge theory is

$$b^i = \frac{1}{2}(b_B + b_C + b_D) = -\frac{10}{3} C(O_{16}) + \frac{4}{3} C(S_{16}) = \frac{76}{3}. \quad (5.29)$$

Even if in this example the gauge theory is non-asymptotically free, it remains a good exercise that illustrates the sector by sector analysis of the gauge threshold corrections.

Here also, the agreement with the direct evaluation of the  $\beta$ -function coefficient of the  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  theory can be checked. This can be done in two steps. At the  $\mathcal{N} = 2$  level obtained by applying the  $\mathbb{Z}_2$  action that breaks spontaneously  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ , the massless spectrum contains an  $\mathcal{N} = 2$  vector multiplet in the adjoint representation of  $G^i = SO(16)$ , coupled to a hypermultiplet in the spinorial representation. Applying the final  $\mathbb{Z}_2^{\text{shift}}$  responsible for the  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  spontaneous breaking, the massless spectrum charged under the  $G^i = SO(16)$  gauge group factor are the bosons of the  $\mathcal{N} = 2$  vector multiplet in the adjoint representation of  $SO(16)$ , together with the fermions of the hypermultiplet in the spinorial representation. Consistently, one finds

$$b^i = \left(-\frac{11}{3} + \frac{2}{6}\right) C(O_{16}) + \frac{4}{3} C(S_{16}) = -\frac{10}{3} C(O_{16}) + \frac{4}{3} C(S_{16}). \quad (5.30)$$

### 5.3. Example 3: Gauge group factor $SO(8) \times SO(8)'$ with $n_F \neq 0$

The third example we would like to present has a  $G^i = SO(8) \times SO(8)'$  gauge subgroup. It is obtained by coupling non-trivially both  $(g, h)$  and  $(G, H)$ , with the vectorial and spinorial representations of  $SO(8) \times SO(8)'$  initially present in the  $\bar{E}_8$  character:

$$\begin{aligned} Z_{4,20} \left[ \begin{smallmatrix} h & H \\ g & G \end{smallmatrix} \right] &= Z_{4,12} \left[ \begin{smallmatrix} h & H \\ g & G \end{smallmatrix} \right] Z_{0,8} \left[ \begin{smallmatrix} h & H \\ g & G \end{smallmatrix} \right] \quad \text{where} \\ Z_{0,8} \left[ \begin{smallmatrix} h & H \\ g & G \end{smallmatrix} \right] &= \frac{1}{2} \sum_{\gamma, \delta} \frac{\bar{\theta}_\delta^4 [\gamma] \bar{\theta}_{\delta+G}^4 [\gamma+H]}{\bar{\eta}^8} e^{i\pi(g\gamma+h\delta+hg+GH)}. \end{aligned} \quad (5.31)$$

As in Example 2, the coupling to  $(h, g)$  breaks  $E_8 \rightarrow SO(16)$  and supersymmetry to  $\mathcal{N} = 0$ , while the coupling to  $(H, G)$  breaks further  $SO(16) \rightarrow SO(8) \times SO(8)'$ . Here also, the SSS phase is effectively

$$S = e^{i\pi[g(a+\gamma)+h(b+\delta)]}. \quad (5.32)$$

Since  $(H, G) = (0, 0)$  in sector  $B$ , the latter is identical to that of Example 2. Therefore, we have  $G_B^i = SO(16)$ , with  $\beta$ -function coefficient

$$b_B^i = -\frac{8}{3} \{C(O_{16}) - C(S_{16})\}. \quad (5.33)$$

However, since the overall gauge group factor of the model is  $G^i = SO(8) \times SO(8)'$ , it is instructive to express the characters of  $G_B^i = SO(16)$  in terms of those of  $SO(8) \times SO(8)'$ :

$$\bar{O}_{16} = \bar{O}_8 \bar{O}'_8 + \bar{V}_8 \bar{V}'_8, \quad \bar{S}_{16} = \bar{S}_8 \bar{S}'_8 + \bar{C}_8 \bar{C}'_8. \quad (5.34)$$

Thus, the bosons of the initially massless  $\mathcal{N} = 4$  vector multiplets in the  $\bar{O}_{16}$  representation (*i.e.* for  $\gamma = 0$ ) are in the adjoint representation  $(\mathbf{28}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{28})$  as well as in the bi-vectorial  $(\mathbf{8}_v, \mathbf{8}_v)$  of  $SO(8) \times SO(8)'$ . Moreover, the fermions of the initially massless  $\mathcal{N} = 4$  vector multiplets in the  $\bar{S}_{16}$  representation (*i.e.* for  $\gamma = 1$ ) are in the  $(\mathbf{8}_s, \mathbf{8}_s)$  and  $(\mathbf{8}_c, \mathbf{8}_c)$  bi-spinorial representations of  $SO(8) \times SO(8)'$ .

As said before, the model can be constructed by successive breaking,

$$E_8 \rightarrow SO(16) \rightarrow SO(8) \times SO(8)', \quad (5.35)$$

by first coupling the  $SO(8) \times SO(8)'$  characters initially present in  $\bar{E}_8$ ,

$$\bar{E}_8 = \bar{O}_{16} + \bar{S}_{16} = \bar{O}_8 \bar{O}'_8 + \bar{V}_8 \bar{V}'_8 + \bar{S}_8 \bar{S}'_8 + \bar{C}_8 \bar{C}'_8, \quad (5.36)$$

with  $(h, g)$ , and then with  $(H, G)$ . In the intermediate step, which is nothing but the sector  $B$ , the  $G_B^i = SO(16)$  gauge theory is non-supersymmetric. However, the analysis of the sectors  $C$  and  $D$  is more conveniently done by considering the model from two other viewpoints:

- The breaking (5.35) can be realized by first coupling the  $SO(8) \times SO(8)'$  characters with  $(H, G)$ , and then with  $(h, g)$ . In the intermediate step, which is nothing but the sector  $C$ , we have an  $\mathcal{N}_C = 2$  supersymmetric  $G_C^i = SO(16)$  gauge theory.
- The breaking (5.35) can also be realized by first coupling the  $SO(8) \times SO(8)'$  characters with  $(h, g) = (H, G)$ , and then with  $(h - H, g - G)$ . In the intermediate step, which is nothing but the sector  $D$ , we have an  $\mathcal{N}_D = 2$  supersymmetric  $G_D^i = SO(16)$  gauge theory.

Actually, the three intermediate gauge group factors  $G_{B,C,D}^i = SO(16)$  are not aligned, so that the resulting unbroken gauge group of the combined final theory is  $G^i = SO(8) \times SO(8)'$ . Correspondingly, thanks to the triality symmetry of the three  $SO(8)$  representations  $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c$ , there are three alternative decompositions of the  $SO(16)$  characters in terms of  $SO(8) \times SO(8)'$  ones. If desired, these decompositions can be used to describe the spectra in sectors  $B, C, D$  in terms of  $SO(8) \times SO(8)'$  representations. They are

$$\begin{aligned} \text{in sector } B: \quad & \bar{O}_{16} = \bar{O}_8 \bar{O}'_8 + \bar{V}_8 \bar{V}'_8, \quad \bar{S}_{16} = \bar{S}_8 \bar{S}'_8 + \bar{C}_8 \bar{C}'_8, \\ \text{in sector } C: \quad & \bar{O}_{16} = \bar{O}_8 \bar{O}'_8 + \bar{S}_8 \bar{S}'_8, \quad \bar{S}_{16} = \bar{C}_8 \bar{C}'_8 + \bar{V}_8 \bar{V}'_8, \\ \text{in sector } D: \quad & \bar{O}_{16} = \bar{O}_8 \bar{O}'_8 + \bar{C}_8 \bar{C}'_8, \quad \bar{S}_{16} = \bar{V}_8 \bar{V}'_8 + \bar{S}_8 \bar{S}'_8. \end{aligned} \quad (5.37)$$

In any case, what we are interested in is the massless spectrum in sector  $C$ , charged under the gauge group factor  $G_C^i = SO(16)$ . To find it, we start from the parent  $\mathcal{N} = 4$  theory, where the massless spectrum contains an  $\mathcal{N} = 4$  vector multiplet in the adjoint representation of  $E_8$ . Implementing the  $(H, G)$ -projection and using the fact that  $\bar{E}_8 = \bar{O}_{16} + \bar{S}_{16}$ , we obtain the sector  $C$ , whose massless spectrum lies schematically in the representation

$$(\mathcal{N}_C = 2 \text{ vector multiplet}) \cdot \bar{O}_{16} \oplus (\mathcal{N}_C = 2 \text{ hypermultiplet}) \cdot \bar{S}_{16}. \quad (5.38)$$

We have an  $\mathcal{N}_C = 2$  vector multiplet in the adjoint representation and a hypermultiplet in the spinorial representation, so that

$$b_C^i = -2\{C(O_{16}) - C(S_{16})\}. \quad (5.39)$$

By symmetry between the sectors  $C$  and  $D$ , we also have in sector  $D$  for the gauge group factor  $G_D^i = SO(16)$ ,

$$b_D^i = -2 \{C(O_{16}) - C(S_{16})\}. \quad (5.40)$$

Combining the above results, the  $\beta$ -function coefficient of the  $G^i = SO(8) \times SO(8)'$  non-supersymmetric gauge theory is

$$b^i = \frac{1}{2}(b_B + b_C + b_D) = -\frac{10}{3} \{C(O_{16}) - C(S_{16})\} = \frac{20}{3}. \quad (5.41)$$

Here also, the gauge theory is non-asymptotically free.

To check the above value of  $b^i$ , we can derive the massless spectrum of the theory that is charged under  $G^i = SO(8) \times SO(8)'$ . We have just seen that the implementation of the  $(H, G)$ -projection on the parent  $\mathcal{N} = 4$  theory leads to the massless spectrum of sector  $C$ , given in Eq. (5.38). Using the decomposition of the  $SO(16)$  characters in terms of  $SO(8) \times SO(8)'$  ones valid in sector  $C$ , this spectrum can be written as

$$(\mathcal{N}_C = 2 \text{ vector multiplet}) \cdot (\bar{O}_8 \bar{O}'_8 + \bar{S}_8 \bar{S}'_8) \oplus (\mathcal{N}_C = 2 \text{ hypermultiplet}) \cdot (\bar{C}_8 \bar{C}'_8 + \bar{V}_8 \bar{V}'_8). \quad (5.42)$$

We can now implement the final  $(h, g)$ -projection, which let us with massless states schematically as follows:

$$\begin{aligned} &(\text{bosons of the vector multiplet}) \cdot \bar{O}_8 \bar{O}'_8 \oplus (\text{fermions of the vector multiplet}) \cdot \bar{S}_8 \bar{S}'_8 \oplus \\ &(\text{bosons of the hypermultiplet}) \cdot \bar{V}_8 \bar{V}'_8 \oplus (\text{fermions of the hypermultiplet}) \cdot \bar{C}_8 \bar{C}'_8. \end{aligned} \quad (5.43)$$

We have 1 gauge boson and 2 real scalars in the adjoint representation of  $SO(8) \times SO(8)'$ ,  $(\mathbf{28}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{28})$ , together with 4 real scalars in the  $(\mathbf{8}_v, \mathbf{8}_v)$ , and 4 Majorana fermions in the  $(\mathbf{8}_s, \mathbf{8}_s) \oplus (\mathbf{8}_c, \mathbf{8}_c)$ . Since the gauge coupling of  $G^i$  is equal to that of each of its  $SO(8)$  subgroups, it is sufficient to calculate the  $\beta$ -function coefficient associated to one of them:

$$b^i = \left(-\frac{11}{3} + \frac{2}{6}\right) C(O_8) + \frac{4n(V'_8)}{6} C(V_8) + \frac{4n(S'_8)}{3} C(S_8) + \frac{4n(C'_8)}{3} C(C_8), \quad (5.44)$$

where  $C(O_8) = 6$ ,  $C(V_8) = C(S_8) = C(C_8) := C_{(8)} = 1$  and the multiplicities arising from the second  $SO(8)'$  factor are all equal,  $n(V'_8) = n(S'_8) = n(C'_8) = 8$ . In total, one has

$$b^i = -\frac{10}{3} C(O_8) + \frac{80}{3} C_{(8)} = \frac{20}{3}, \quad (5.45)$$

which is in agreement with the sector by sector contributions.

#### 5.4. The generic case

The above examples illustrate the *universal structure* of the running effective gauge couplings valid in the  $\mathbb{Z}_2$  non-symmetric orbifold models that realize a spontaneous  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  supersymmetry breaking à la SSS, when shifts (but no dual shifts) are introduced along the untwisted torus. In these models, no dangerous linear dependence on the internal volume appears in the threshold corrections. The result is given in Eq. (5.5) for  $Q < M_{B,C,D}$  (or Eq. (5.16) for  $Q < cM_s$ ), with the sector by sector  $\beta$ -function coefficients given by:

$$\begin{aligned}
b_B^i &= -\frac{8}{3} \{C(O_B) - C(\mathcal{R}_B)\}, \quad b_C^i = -2 \{C(O_C) - C(\mathcal{R}_C)\}, \\
b_D^i &= -2 \{C(O_D) - C(\mathcal{R}_D)\}.
\end{aligned} \tag{5.46}$$

The structures of the sectors  $C$  and  $D$  are simple to understand, since both of them describe  $\mathcal{N} = 2$  supersymmetric gauge theories. The associated gauge groups contain factors  $G_C^i$  and  $G_D^i$ , which may be different. The individual  $\beta$ -function coefficients are given in terms of vector multiplets contributions in the adjoint representations of  $G_{C,D}^i$ , denoted by  $-2C(O_{C,D})$ , together with hypermultiplets contributions in the representations  $\mathcal{R}_{C,D}$ , denoted by  $2C(\mathcal{R}_{C,D})$ .

On the contrary, the structure of sector  $B$ , which describes a non-supersymmetric gauge theory with a gauge group factor  $G_B^i$ , is something new. The  $-\frac{8}{3}C(O_B)$  contribution to  $b_B^i$  comes from the bosons of initially massless  $\mathcal{N} = 4$  vector multiplets in the parent  $\mathcal{N} = 4$  model, that remain massless. These bosons (1 vector and 2 real scalars) are in the adjoint representation of  $G_B^i$ . The second contribution,  $\frac{8}{3}C(\mathcal{R}_B)$ , arises from the fermions of initially massless  $\mathcal{N} = 4$  vector multiplets in the parent theory, that remain massless. They are 4 Majorana fermions in a spinorial representation  $\mathcal{R}_B$ . If as in Examples 2 and 3,  $\mathcal{R}_B$  is a spinorial representation of  $SO(16)$ , it is in general the spinorial representation of a subgroup of  $E_8$ , such as  $SO(16)$ ,  $SO(8) \times SO(8)'$ ,  $E_7 \times SU(2)$ ,  $SO(12) \times SO(4)$  or even  $SO(4)^4$ . All these cases can be easily realized by fermionic constructions.

## 6. 2nd plane, 3rd plane and $\mathcal{N} = 1$ sector contributions: $(H_2, G_2) \neq (0, 0)$

In Sections 4 and 5, we have extensively analyzed the threshold corrections in  $\mathbb{Z}_2$  non-symmetric orbifold models, where an  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \rightarrow \mathcal{N} = 0$  spontaneous breaking of supersymmetry is implemented with shifts such that the running gauge couplings develop only logarithmic dependencies on the volume of the untwisted internal 2-torus. Up to an additional overall factor of  $\frac{1}{2}$ , these results are the contributions of the  $\mathcal{N} = 4$  and 1st complex plane in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifold models. In the present section, we proceed with the evaluations of the contributions arising from the remaining sectors, namely the 2nd and 3rd complex planes, and the  $\mathcal{N} = 1$  sector. All of them are twisted, with  $(H_2, G_2) \neq (0, 0)$ . Moreover, the 2nd plane has  $(H_1, G_1) = (0, 0)$ , the 3rd plane has  $(H_3, G_3) = (0, 0)$  and the  $\mathcal{N} = 1$  sector has  $(H_1, G_1) \neq (0, 0)$ ,  $(H_1, G_1) \neq (H_2, G_2)$ .

Our concern in the present paper is the decompactification problem only. In particular, we do not address the issue of chirality and the models presented here are actually incompatible with the physical requirement that the  $\mathcal{N} = 1$  spectrum (further spontaneously broken to  $\mathcal{N} = 0$  à la SSS) be chiral. Forgetting for the moment the final breaking to  $\mathcal{N} = 0$ , we remind that at the level of  $\mathcal{N} = 1$  supersymmetric models constructed via  $\mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric moduli-deformed fermionic construction, the chiral families always come from the  $\mathcal{N} = 2$  twisted sectors that have non-trivial fixed points. Thus, the  $\mathcal{N} = 1$  untwisted spectrum ( $H_1 = H_2 = 0$  projected by  $G_1, G_2$ ) is always non-chiral, while the spectrum arising in the 1st plane ( $H_1 = 1, H_2 = 0$  projected by  $G_1, G_2$ ) is massive, when the 1st  $\mathbb{Z}_2$  action acts freely so that no fixed point arises in this plane. To understand why the spectrum realized in the 2nd and 3rd planes is also non-chiral in this case, we reverse the role of the two  $\mathbb{Z}_2$  actions. The freely acting  $\mathbb{Z}_2$  can be viewed as responsible of an  $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$  spontaneous breaking of supersymmetry on a  $T^6/\mathbb{Z}_2$  parent model. This means that in the large volume limit of the 1st internal 2-torus, one recovers an  $\mathcal{N} = 2$  spectrum. However, in the 2nd and 3rd complex planes, the 1st torus, which is shifted, is also twisted since  $H_2 = 1$ . The spectrum arising from these sectors is thus independent of the moduli  $T_1, U_1$  and is

identical to the non-chiral one present in the large volume limit, where  $\mathcal{N} = 2$  is recovered. In the case of  $(2, 2)$  compactifications, which correspond to Calabi–Yau internal spaces at fermionic points, the Euler characteristic vanishes [28].

Taking into account the final breaking of  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ , we have in the partition function (see Appendix A and Ref. [15])

$$Z_{2,2} \left[ \begin{smallmatrix} h_1^1, h_1^2 \\ g_1^1, g_1^2 \end{smallmatrix} \middle| \begin{smallmatrix} H_2 \\ G_2 \end{smallmatrix} \right] = \begin{cases} \frac{\Gamma_{2,2} \left[ \begin{smallmatrix} h_1^1, h_1^2 \\ g_1^1, g_1^2 \end{smallmatrix} \right] (T_1, U_1)}{(\eta\bar{\eta})^2}, & \text{when } (H_2, G_2) = (0, 0), \\ \frac{4\eta\bar{\eta}}{\theta \left[ \begin{smallmatrix} 1-H_2 \\ 1-G_2 \end{smallmatrix} \right] \bar{\theta} \left[ \begin{smallmatrix} 1-H_2 \\ 1-G_2 \end{smallmatrix} \right]} \delta \left[ \begin{smallmatrix} h_1^1 & H_2 \\ g_1^1 & G_2 \end{smallmatrix} \right]_{0 \bmod 2} \delta \left[ \begin{smallmatrix} h_2^1 & H_2 \\ g_2^1 & G_2 \end{smallmatrix} \right]_{0 \bmod 2}, & \text{when } (H_2, G_2) \neq (0, 0), \end{cases} \quad (6.1)$$

where the shifts  $(h_1^i, g_1^i)$  are defined in Eq. (4.5) (we remind that  $H, G$  denote  $H_1, G_1$ ). Therefore, the twisted spectrum of the 2nd and 3rd complex planes ( $H_2 = 1$  projected by  $G_1, G_2$ ) is independent of the gravitino mass  $m_{\frac{3}{2}}^{(E)} \equiv m_B/g_s$ , which is in the desired 1–10 TeV region. The contributions of these modes to the partition function are identical to those evaluated at the fermionic point. We proceed by arguing that the SSS phase in the sectors  $(H_2, G_2) \neq (0, 0)$  must not break supersymmetry to  $\mathcal{N} = 0$ . The reason for this comes in three steps. First, in the 2nd and 3rd planes, the sub-sectors with  $(h, g) = (0, 0)$  always preserve  $\mathcal{N} = 2$  supersymmetry, and since we choose to restrict to the case where they arise from non-free  $\mathbb{Z}_2$  actions, the order of magnitude of the moduli  $T_I, U_I$ ,  $I = 2, 3$ , must be close enough to 1 for the decompactification problem not to arise. Second, if the  $(h, g) \neq (0, 0)$  sub-sectors of the 2nd and 3rd planes were non-supersymmetric, the respective gravitini mass scales would be determined by  $T_I, U_I$ ,  $I = 2, 3$ , and thus of order  $M_{\text{Planck}}$ , when measured in Einstein frame, which is something we want to exclude. Third, the  $(h, g) \neq (0, 0)$  sub-sector of the  $\mathcal{N} = 1$  sector must preserve supersymmetry as well, in order to not lead to an extremely large gravitino mass. To summarize, in our solution to the decompactification problem, the SSS phase  $S$  in the sectors  $(H_2, G_2) \neq (0, 0)$  must not contain the factor  $e^{i\pi(ag+bh+hg)}$  introduced in Eq. (4.4), which would otherwise break susy to  $\mathcal{N} = 0$  at tree level at a high scale. The breaking of supersymmetry is transmitted to the 2nd and 3rd planes twisted spectra ( $H_2 = 1$  projected by  $G_1, G_2$ ) by quantum corrections that involve states with broken supersymmetry ( $H_2 = 0$  projected by  $G_1, G_2$ ). To summarize, the spectrum arising from the 2nd and 3rd planes presents at tree level an  $\mathcal{N} = 2$  extended supersymmetry and is non-chiral.

Note that since the sectors  $(H_2, G_2) = (0, 0)$  and  $(H_2, G_2) \neq (0, 0)$  are independent orbits of the worldsheet modular group, the associated choices of SSS phases do not need to be correlated to guaranty the consistency of the whole  $\mathbb{Z}_2 \times \mathbb{Z}_2$  model. In the sectors  $(H_2, G_2) \neq (0, 0)$ , a certainly valid susy preserving choice is  $S \equiv 1$ . However, playing with the quantum numbers  $(H_1, G_1)$  and  $(H_2, G_2)$ , we can have

$$\text{In the sectors } (H_2, G_2) \neq (0, 0), \quad S = e^{i\pi[\zeta_1(aG_1+bH_1+H_1G_1)+\zeta_2(aG_2+bH_2+H_2G_2)]}, \quad (6.2)$$

where  $\zeta_1$  and  $\zeta_2$  can be fixed to 0 or 1. As we just noticed,  $\zeta_1$  may not be equal to  $\zeta'$  we introduced in Eq. (4.4). To see that  $(\zeta_1, \zeta_2) = (0, 0)$  is not the only allowed choice, we consider the conformal block associated to the left-moving fermionic degrees of freedom,

$$\begin{aligned}
& \frac{1}{2} \sum_{a,b} e^{i\pi(a+b+ab)} e^{i\pi[\zeta_1(aG_1+bH_1+H_1G_1)+\zeta_2(aG_2+bH_2+H_2G_2)]} \\
& \quad \times \theta_b^a(2v) \theta_{b+G_2}^{a+H_2} \theta_{b+G_1}^{a+H_1} \theta_{b-G_1-G_2}^{a-H_1-H_2} \\
& = e^{i\pi(\zeta_1+\zeta_2)(H_1G_2-G_1H_2)} e^{i\pi(G_1+G_2)(1+H_1+H_2)} \theta_1^1(v) \theta_{1-G_2}^{1-H_2}(v) \theta_{1-G_1}^{1-H_1}(v) \\
& \quad \times \theta_{1+G_1+G_2}^{1+H_1+H_2}(v). \tag{6.3}
\end{aligned}$$

To show this equality, one can redefine  $a = A - \zeta_1 H_1 - \zeta_2 H_2$ ,  $b = B - \zeta_1 G_1 - \zeta_2 G_2$  and sum over  $A, B$  equal to 0 or 1. Given that  $(H_2, G_2) \neq (0, 0)$ , we see that  $\mathcal{N} = 2$  supersymmetry is preserved in the 2nd plane,  $(H_1, G_1) = (0, 0)$ , and in the 3rd plane,  $(H_1, G_1) = (H_2, G_2)$ , (or  $(H_3, G_3) = (0, 0)$ ). Supersymmetry is also preserved in the  $\mathcal{N} = 1$  sector,  $|\frac{H_1}{G_1} \frac{H_2}{G_2}| \neq 0$ . Two distinct cases arise however,  $\zeta_1 = \zeta_2$  or  $\zeta_1 = 1 - \zeta_2$ , corresponding to different choices of discrete torsions that yield opposite contributions of the  $\mathcal{N} = 1$  sector to the partition function.

The  $\mathcal{N}_I = 2$ ,  $I = 2, 3$ , unbroken supersymmetries of the 2nd and 3rd planes are *not aligned* to one another, as well as *non-aligned* with the  $\mathcal{N}_C = 2$  and  $\mathcal{N}_D = 2$  supersymmetries appearing in the sectors  $C$  and  $D$  of the 1st complex plane. Being supersymmetric, the 2nd plane, 3rd plane and  $\mathcal{N} = 1$  sector do not contribute to the effective potential. Moreover, their contributions to the gauge coupling thresholds are identical to those present in the  $\mathcal{N} = 1$  supersymmetric  $\mathbb{Z}_2 \times \mathbb{Z}_2$  moduli-deformed fermionic models. In this class of theories, the  $\mathcal{N} = 1$  sectors do not contribute. The reason for this is that the helicity operator  $Q_b^a$  acting on an  $\mathcal{N} = 1$  sector involves

$$\begin{aligned}
& \partial_v^2 \left( \theta_1^1(v) \theta_{1-G_2}^{1-H_2}(v) \theta_{1-G_1}^{1-H_1}(v) \theta_{1+G_1+G_2}^{1+H_1+H_2}(v) \right) \Big|_{v=0} \\
& \propto \partial_v^2 \left( \theta_1(v) \theta_2(v) \theta_3(v) \theta_4(v) \right) \Big|_{v=0} = 0, \tag{6.4}
\end{aligned}$$

thanks to the fact that  $\theta_1(v)$  is odd and  $\theta_{2,3,4}(v)$  are even. Therefore, corrections to the gauge couplings occur only from the  $\mathcal{N} = 2$  planes. The case of  $\mathcal{N} = 2$  planes in symmetric orbifolds, which are characterized by  $(2, 2)$  superconformal symmetry, have been analyzed extensively in the literature [18]. However, even if the analysis for non-symmetric orbifolds that possess  $(2, 0)$  superconformal symmetry has not yet been fully completed, our conclusions will remain valid in this case, as mentioned later in this section.

Let us start by considering the 2nd and 3rd planes in the  $(2, 2)$  case. As was shown in Refs [18, 19, 21, 29], the gauge coupling corrections are given in terms of two threshold functions,

$$\Delta_I^i = b_I^i \Delta(T_I, U_I) - k^i Y(T_I, U_I), \quad I = 2, 3, \tag{6.5}$$

where  $b_I^i$  are the  $\mathcal{N} = 2$   $\beta$ -function coefficients in each planes,<sup>9</sup>

$$\begin{aligned}
\Delta(T_I, U_I) &= -\log \left( 4\pi^2 |\eta(T_I)|^4 |\eta(U_I)|^4 \text{Im } T_I \text{Im } U_I \right), \\
Y(T_I, U_I) &= -\frac{\xi}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T_I, U_I) \left[ \left( \bar{E}_2 - \frac{3}{\pi \tau_2} \right) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right]. \tag{6.6}
\end{aligned}$$

In these expressions,  $E_{2,4,6}$  are holomorphic Eisenstein series, with modular weights 2, 4, 6,

<sup>9</sup> In our conventions,  $b_I^i$ ,  $I = 2, 3$ , are  $\beta$ -function coefficients in the parent theories obtained by acting with a single  $\mathbb{Z}_2$ . In the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models we are interested in, overall factors  $\frac{1}{2}$  must be included in the r.h.s. of Eq. (6.5), for the thresholds to be correctly normalized.

$$\begin{aligned}
E_2 &= \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \\
E_4 &= \frac{1}{2} (\theta_2^8 + \theta_3^8 + \theta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \\
E_6 &= \frac{1}{2} (\theta_2^4 + \theta_3^4) (\theta_3^4 + \theta_4^4) (\theta_4^4 - \theta_2^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}, \tag{6.7}
\end{aligned}$$

while  $j = \frac{1}{q} + 744 + \mathcal{O}(q)$  is holomorphic and modular invariant.  $\xi$  is a constant that can be expressed in terms of the numbers of massless vector multiplets and hypermultiplets per plane. Using the relation between gauge and  $\mathcal{R}^2$ -term renormalizations [15], it is fixed to  $\xi = -1$ , thanks to the anomaly cancellation conditions [30] valid in the six dimensional decompactification limits [15,18]. This property being general in all  $\mathcal{N} = 2$  theories with underlying  $(2, 2)$  superconformal symmetries, the threshold corrections are universal in this case [21,27], modulo the  $\beta$ -function coefficients and Kac–Moody levels.

As anticipated, what is relevant to note is that these threshold corrections scale linearly with the volume of the untwisted 2-tori. For  $\text{Im } T_I \gg 1$  and  $U_I$  finite, one has

$$\Delta(T_I, U_I) = \frac{\pi}{3} \text{Im } T_I - \log(\text{Im } T_I) + \mathcal{O}(1), \quad Y(T_I, U_I) = 4\pi \text{Im } T_I + \mathcal{O}\left(\frac{1}{\text{Im } T_I}\right), \tag{6.8}$$

which invalidates the string perturbative expansion (when the dressing  $\beta$ -function coefficient is negative). As follows from target space duality, similar dangerous behaviors occur in all limits, where the Kähler and/or complex structures of the untwisted 2-tori are large or small:  $T_I \rightarrow \infty$  or 0, and/or  $U_I \rightarrow \infty$  or 0. This is not a surprise, since we have seen in the previous sections (and also in Ref. [18]) that for the linear terms not to arise,  $\mathcal{N} = 4$  supersymmetry must be restored on the moduli space boundary. However, this cannot be the case in our 2nd and 3rd complex planes, since the breaking from  $\mathcal{N} = 4$  to  $\mathcal{N} = 2$  in these sectors is not spontaneous. As announced before in this section, these considerations force us to assume that the order of magnitude of the moduli of the 2nd and 3rd planes,  $T_I$  and  $U_I$ ,  $I = 2, 3$ , are not too far from 1. This justifies that we took the order of magnitude of the coefficient  $c$  introduced in Eq. (4.18) to be not far from 1. Moreover, the moduli-dependent scales  $M_I$ 's that control the threshold corrections are

$$\begin{aligned}
\frac{1}{M_I^2} &= \frac{16}{M_s^2} |\eta(T_I)|^4 |\eta(U_I)|^4 \text{Im } T_I \text{Im } U_I, \quad I = 2, 3 \\
&= \frac{16}{M_{\text{Planck}}^2} |\eta(T_I)|^4 |\eta(U_I)|^4 \text{Im } S_{\text{dil}} \text{Im } T_I \text{Im } U_I, \tag{6.9}
\end{aligned}$$

and are close to the string scale  $M_s$ . In the above expression, we introduce the string coupling constant, which is related to the dilaton field,  $g_s^2 = 1/\text{Im } S_{\text{dil}}$ , in order to display the threshold masses in units of gravitational scale.

The contributions  $b_I^j \Delta(T_I, U_I)$  controlled by the  $M_I$ 's have to be completed by the universal contribution  $-k_I Y(T_I, U_I)$ , whose order of magnitude is close to 1. Being infrared finite, these corrections are continuous functions that remain finite even at special values of  $(T_I, U_I)$ , where



additional massless states arise. Thus, we are free to absorb them in a redefinition of the string coupling [29]:

$$\frac{16\pi^2}{g_{\text{renor}}^2} = \frac{16\pi^2}{g_s^2} - \frac{1}{2}Y(T_2, U_2) - \frac{1}{2}Y(T_3, U_3), \quad (6.10)$$

where the factors  $\frac{1}{2}$  arise from the action of the second  $\mathbb{Z}_2$  (see footnote 9) and the “renormalized” string coupling is

$$g_{\text{renor}}^2 = \frac{g_s^2}{1 - \frac{1}{32\pi^2}(Y(T_2, U_2) + Y(T_3, U_3))g_s^2}. \quad (6.11)$$

When the 2nd and 3rd complex planes are realized as  $(2, 0)$  non-symmetric compactifications *via* fermionic constructions, the natural values for  $\text{Im } T_I$  and  $\text{Im } U_I$  are of order 1. Moreover, the target space dualities  $SL(2, \mathbb{Z})_{T_I} \times SL(2, \mathbb{Z})_{U_I}$  of the  $(2, 2)$  case are broken to some sub-groups. Consequently,  $|\eta(T_I)|^4$  and  $|\eta(U_I)|^4$  are replaced by products of other modular functions, with however identical weights. In all cases,  $(2, 2)$  and  $(2, 0)$ , the orders of magnitude of the dressed threshold scales  $M_I$ ,  $I = 2, 3$ , remain close to the string scale.

We are now ready to collect all our previous results and present the 1-loop effective potential and running gauge couplings arising in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  moduli-deformed fermionic construction. We consider models where  $\mathcal{N} = 1$  supersymmetry is further spontaneously broken to  $\mathcal{N} = 0$  at a scale in the TeV regime,  $m_{\frac{3}{2}}^{(\text{E})} = \mathcal{O}(1\text{--}10)$  TeV, while the validity of perturbation theory is preserved. Our work is restricted to the case where only the 1st  $\mathbb{Z}_2$  action is free. The second one and the product of the two have fixed points. Under these conditions, only one internal 2-torus, the 1st in our conventions, is large and involved in the  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  breaking and  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  breaking, which are both spontaneous. This was done by introducing suitable shifts along this torus but dual shifts may have been considered.

In these models, we find remarkable that the  $\mathcal{N} = 4$  sector spontaneously broken to  $\mathcal{N} = 0$ , which is referred as sector  $B$ , is the only one leading to a substantial contribution to the effective potential (the cosmological term), when  $m_B \equiv m_{\frac{3}{2}}$  is small compared to  $cM_s$ ,

$$\begin{aligned} V_{\text{eff}} &= \frac{1}{4}V_{\text{eff } B} + \mathcal{O}\left(e^{-c\sqrt{\text{Im } T_1}}\right) \\ &= -\frac{1}{4}\frac{2 + d_{G_B} - n_{F_B}}{2\pi^7}\frac{1}{(\text{Im } T_1)^2}E_{\alpha_B, \beta_B}(U_1|3) + \mathcal{O}\left(e^{-c\sqrt{\text{Im } T_1}}\right), \end{aligned} \quad (6.12)$$

which is proportional to  $m_{\frac{3}{2}}^4$ . Moreover, the relevant threshold corrections to the gauge couplings arise from the sector  $B$ , as well as from four sectors exhibiting exact  $\mathcal{N} = 2$  supersymmetries: The sectors  $C$  and  $D$ , which are actually sub-sectors of the “massive” 1st complex plane, and the 2nd and 3rd complex planes. The associated  $\mathcal{N}_C, \mathcal{N}_D, \mathcal{N}_2, \mathcal{N}_3 = 2$  supersymmetries are all non-aligned. These five contributions to the gauge coupling thresholds are characterized by effective mass scales:  $M_{B,C,D}$  depend on the “massive” 1st plane moduli  $T_1, U_1$ , while  $M_I$ ,  $I = 2, 3$ , depends on the  $I$ th plane moduli  $T_I, U_I$  and is modular invariant, with respect to some target space duality sub-group of  $SL(2, \mathbb{Z})_{T_I} \times SL(2, \mathbb{Z})_{U_I}$ .

The running effective coupling constants in the  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  models take a very simple form, once expressed in terms of the dressed mass scales and coupling  $g_{\text{renor}}$ ,

$$\frac{16\pi^2}{g_i^2(Q)} = k^i \frac{16\pi^2}{g_{\text{renor}}^2} - \frac{1}{4}b_B^i \log\left(\frac{Q^2}{Q^2 + M_B^2}\right) - \frac{1}{4}b_C^i \log\left(\frac{Q^2}{Q^2 + M_C^2}\right)$$

$$-\frac{1}{4}b_D^i \log\left(\frac{Q^2}{Q^2 + M_D^2}\right) - \frac{1}{2}b_2^i \log\left(\frac{Q^2}{M_2^2}\right) - \frac{1}{2}b_3^i \log\left(\frac{Q^2}{M_3^2}\right), \quad (6.13)$$

where  $Q < cM_s$  is the energy scale measured in string frame ( $Q^{(E)} < cM_{\text{Planck}}$  in the Einstein frame) and the sector by sector  $\beta$ -coefficients are

$$\begin{aligned} b_B^i &= -\frac{8}{3}\{C(O_B) - C(\mathcal{R}_B)\}, \quad b_C^i = -2\{C(O_C) - C(\mathcal{R}_C)\}, \\ b_D^i &= -2\{C(O_D) - C(\mathcal{R}_D)\}, \quad b_2^i = -2\{C(O_2) - C(\mathcal{R}_2)\}, \\ b_3^i &= -2\{C(O_3) - C(\mathcal{R}_3)\}. \end{aligned} \quad (6.14)$$

The  $O_{B,C,D,2,3}$  and  $\mathcal{R}_{B,C,D,2,3}$  symbols refer to adjoint and matter representations of gauge group factors  $G_{B,C,D,2,3}^i$  that are realized in the sectors  $B, C, D$  and  $I = 2, 3$ , respectively. In total, the  $\beta$ -function coefficient of the  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  model, for  $Q$  smaller than all threshold scales, is given by

$$b^i = \frac{1}{4}(b_B^i + b_C^i + b_D^i) + \frac{1}{2}(b_2^i + b_3^i). \quad (6.15)$$

When  $\text{Im } U_1 = \mathcal{O}(1)$ , the dressed masses measured in Einstein frame,  $M_{B,C,D}^{(E)} = M_{B,C,D}/g_s$ , are all in the TeV region. Thus, they decouple in Eq. (6.13), when  $Q^{(E)} = Q/g_s$  reaches larger energy scales, thanks to the restoration of  $\mathcal{N} = 4$  supersymmetry in the sector  $B$  and 1st plane. When  $\text{Im } U_1$  or  $1/\text{Im } U_1$  is larger, say up to 20 or so, only two scales among  $M_{B,C,D}^{(E)}$  are in the TeV region, while the remaining one can be up to  $cM_{\text{Planck}}$ . In this case, the full restoration of  $\mathcal{N} = 4$  supersymmetry in the sector  $B$  and 1st plane occurs only at energies above this highest threshold scale. In Eq. (6.13), the reason why we do not add  $Q^2$ -terms in front of the  $M_I^2$ 's,  $I = 2, 3$ , is that the order of magnitude of these two threshold masses is close to the string scale  $M_s$ , and that in our effective description, the physical energy  $Q$  must not exceed  $cM_s$ .

From the effective field theory viewpoint, the SSS susy breaking gives rise to a specific  $\mathcal{N} = 1$  supergravity no-scale model, with so-called “ $S_{\text{dil}} T_1 U_1$ ”-breaking mechanism [31]. We remind that  $S_{\text{dil}}$  is the four dimensional dilaton, while  $T_1, U_1$  are the moduli of the “massive” 1st complex plane. The moduli of the 2nd and 3rd planes do not participate in the supersymmetry breaking. As explained in Ref. [31], the determination *via* radiative corrections of the vacuum expectation value of the “no-scale modulus” and thus of the  $\mathcal{N} = 1$  gravitino mass  $m_{\frac{3}{2}}^{(E)}$  [32], at relatively low scale of order 1–10 TeV, requires that the genus-1 effective potential is free from terms that scale like  $(m_{\frac{3}{2}}^{(E)})^2 \Lambda^2$ . In such terms,  $\Lambda$  is the cut-off of the effective field theory, which in principle can be as large as  $M_{\text{Planck}}$  or  $M_s$ . Thus, it is remarkable that in the setup we consider in this work to break spontaneously  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$ , such terms are absent, thanks to the underlying  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 0$  supersymmetry breaking structure of the sector  $B$ , which imposes the genus-1 effective potential to scale like  $(m_{\frac{3}{2}}^{(E)})^4$ .

## 7. Conclusions

In this paper, our concern is to implement a low scale spontaneous breaking of supersymmetry in  $\mathcal{N} = 1$  models, while maintaining the validity of gauge coupling perturbation theory. We address this question within the context of  $\mathcal{N} = 1 \mathbb{Z}_2 \times \mathbb{Z}_2$  non-symmetric orbifolds, realized by moduli-deformed fermionic constructions. At the  $\mathcal{N} = 1$  supersymmetric level, it is known that

an  $\mathcal{N} = 2$  complex plane realized as an  $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$  spontaneous breaking of supersymmetry yields threshold corrections to the gauge couplings, with a mild logarithmic dependence on the complex plane volume [18]. This contrasts with the case where the  $\mathbb{Z}_2$  action responsible of the  $\mathcal{N} = 4$  breaking to  $\mathcal{N} = 2$  is not freely acting. Indeed, a linear dependence of the thresholds on the complex plane volume arises in this case, invalidating perturbation theory once the volume is large. What we have shown in the present work is that the above solution to the “decompactification problem” can be extended to the case where  $\mathcal{N} = 1$  supersymmetry is further spontaneously broken to  $\mathcal{N} = 0$  at a low scale, by implementing an additional  $\mathbb{Z}_2^{\text{shift}}$  orbifold shift acting along the large internal dimensions and coupled with the helicity charges  $(a, b)$ .

To arrive at this conclusion, we develop a sector by sector analysis of the models and analyze systematically the associated induced threshold corrections. We find that one of the  $\mathbb{Z}_2$  twists, which for instance preserves the 1st complex plane, must act freely. Restricting to the case where no “dual shifts” are implemented along this plane, the  $\mathbb{Z}_2$  twist acts on it as a shift. Allowing the volume of the 1st plane to be large, we can further implement the  $\mathbb{Z}_2^{\text{shift}}$  responsible for the susy breaking to  $\mathcal{N} = 0$  along this plane. As desired, the gravitino mass  $m_{\frac{3}{2}}^{(E)}$  generated this way is low. We find that taking into account the first  $\mathbb{Z}_2$  (which has a free action) and the additional  $\mathbb{Z}_2^{\text{shift}}$  only, three sub-sectors denoted as  $B$ ,  $C$  and  $D$  contribute substantially to the thresholds. What is meant by “substantially” is that other sub-sectors that are non-supersymmetric contribute in the 1st complex plane, but their effects are however exponentially suppressed when the gravitino mass is small,  $m_{\frac{3}{2}}^{(E)} \ll M_{\text{Planck}}$ . Moreover, this hierarchy allows another great simplification, since it implies the contributions of the massive excitations of the string are also exponentially suppressed, compared to those arising from the Kaluza–Klein towers of states above the charged massless states.

The above discussion is general if the 2nd  $\mathbb{Z}_2$  twist and the diagonal product of both  $\mathbb{Z}_2$ ’s have fixed points. In this case, the 2nd and 3rd planes do not arise from a spontaneous breaking of  $\mathcal{N} = 4$  supersymmetry and their volume (in Planck units) and shape moduli must be close to 1, in order not to introduce the decompactification problem back. In addition, supersymmetry has to be preserved at tree level in these sectors, since otherwise an extremely large gravitino mass would be generated. These two planes are the remaining sectors that contribute to the thresholds. Of course, other models where both  $\mathbb{Z}_2$  actions (and eventually their diagonal product as well) are freely acting could be analyzed. In these cases, both the 1st and 2nd (and eventually the 3rd) internal 2-tori are allowed to be large and involved in the spontaneous breaking of the supersymmetries.

In total, the five relevant sectors in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models we consider in the present paper, which have  $\mathcal{N} = 1$  supersymmetry spontaneously broken to  $\mathcal{N} = 0$  at low scale à la SSS, are as follows:

- The sector  $B$ , describes the  $\mathcal{N} = 0$  spontaneously broken phase of the  $\mathcal{N} = 4$  spectrum of the initial parent theory. Surprisingly, *this sector is the only non-supersymmetric one that is relevant for the gauge coupling thresholds and effective potential*. In fact, the other sectors relevant for the gauge couplings being supersymmetric, the sector  $B$  is solely responsible for the generation of the cosmological term. The latter is proportional to  $(m_{\frac{3}{2}}^{(E)})^4$  and no  $M_{\text{Planck}}^2 (m_{\frac{3}{2}}^{(E)})^2$  term is induced.
- The sectors  $C$  and  $D$ , which are both sub-sectors of the non-chiral 1st complex plane, preserve  $\mathcal{N}_C = 2$  and  $\mathcal{N}_D = 2$  supersymmetries, respectively.

- The 2nd and 3rd chiral complex planes preserve  $\mathcal{N}_2 = 2$  and  $\mathcal{N}_3 = 2$  supersymmetries, respectively.

The gauge coupling thresholds arising from the above sectors are controlled by associated mass scales, which are functions of the Kähler and complex structures  $T_I, U_I$  of the corresponding planes,  $I = 1, 2, 3$ . In the 1st plane, the smallest of the masses  $M_B^{(E)}, M_C^{(E)}$  and  $M_D^{(E)}$  is about 1–10 TeV (as is the case for all of them if  $U_1 \simeq i$ ). However, any hierarchy among these scales can be achieved by permuting the formal expressions of  $M_{B,C,D}^{(E)}$ , which can be done by changing the pattern of shifts along the 1st complex plane. On the contrary, in the 2nd and 3rd planes,  $M_2^{(E)}$  and  $M_3^{(E)}$  are close to the Planck scale. Finally, additional universal contributions of order 1 arising from these 2nd and 3rd planes correct slightly the large inverse bare coupling,  $k^i/g_s^2$ .

What we have found is the complete dependence of the running effective gauge couplings on the physical scale  $Q^{(E)}$ , up to  $cM_{\text{Planck}}$ , including when  $Q^{(E)}$  crosses the thresholds scales  $M_{B,C,D}^{(E)}$  and that the associated Kaluza–Klein towers of states decouple from the thresholds. The upper bound  $cM_{\text{Planck}}$ , where  $c$  is not far from 1, is the order of magnitude of the massive string modes in Planck units, whose exponentially suppressed contributions have been neglected. The result, displayed in Eq. (6.13), takes a universal form that depends only on the  $\beta$ -function coefficients associated to the above listed five relevant sectors. Moreover, the form itself of the  $\beta$ -function coefficients is universal, Eqs. (6.14). The factors  $\mp 2$  in the coefficients  $b_C^i, b_D^i, b_2^i, b_3^i$  arise from the massless vector multiplets and hypermultiplets charged under the gauge group factors  $G_{C,D,2,3}^i$ , which are realized in each sectors. The factors  $\mp \frac{8}{3}$  in  $b_B^i$  follow from specific truncations to  $\mathcal{N} = 0$  of the massless  $\mathcal{N} = 4$  vector multiplets in the parent models: 1 vector boson plus 6 real scalars contribute  $-\frac{8}{3}$ , while 4 Majorana fermions contribute  $\frac{8}{3}$ . All these states are charged under a gauge group factor  $G_B^i$ , realized in the sector  $B$ .

While the early examples of realistic free fermionic models consisted in isolated examples [33], in more recent years, systematic classification methods have been developed that enable scanning large classes of three generations models, with viable phenomenological properties [23]. However, in all these vacua [23,33], as well as in other quasi-realistic heterotic string models [34],  $\mathcal{N} = 1$  supersymmetry is unbroken and its spontaneous breaking to  $\mathcal{N} = 0$  needs to be implemented. When this is done *via* Stringy Scherk–Schwarz mechanism in  $\mathbb{Z}_2 \times \mathbb{Z}_2$  fermionic construction, the conditions for the present solution to the decompactification problem to be valid are however incompatible with the physical requirement that the spectrum be chiral (the large volume limit of the 1st internal 2-torus leads to an  $\mathcal{N} = 2$  spectrum and the twisted spectra of the 2nd and 3rd planes are independent of this volume). Thus, implementing an  $\mathcal{N} = 1 \rightarrow \mathcal{N} = 0$  spontaneous breaking of supersymmetry in a realistic, chiral model, while preserving perturbation theory remains a challenge. We also note the recent work of Ref. [35] on the partition functions of non-supersymmetric heterotic string vacua.

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## Appendix A. The moduli-deformed fermionic construction

The context of our study is within the framework of fermionic constructions, where marginal  $(1, 1)$ -current-current deformations are implemented. We restrict to the introduction of the moduli  $T_I$  and  $U_I$ ,  $I = 1, 2, 3$ , associated to the three internal 2-tori involved in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  models in bosonic language. The goal of this appendix is to review the procedure to achieve these deformations. Throughout this paper, our definition for the  $\theta$ -functions is, for  $\alpha, \beta \in \mathbb{R}$ ,

$$\theta_{[\beta]}^{[\alpha]}(v|\tau) = \sum_m q^{\frac{1}{2}(m-\frac{\alpha}{2})^2} e^{2i\pi(v-\frac{\beta}{2})(m-\frac{\alpha}{2})}, \quad \text{where} \quad q = e^{2i\pi\tau}. \quad (\text{A.1})$$

### A.1. One coordinate compactification

In the fermionic construction, one deals with two holomorphic and two antiholomorphic worldsheet Majorana–Weyl fermions  $\omega$ ,  $y$ ,  $\bar{\omega}$ ,  $\bar{y}$ , rather than an internal compactified coordinate  $X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$ . The well known fermion–boson equivalence in two dimensions is established *via* the definitions and identifications

$$\psi = \frac{\omega + iy}{\sqrt{2}} \equiv :e^{i\sqrt{2}X_L}:, \quad \bar{\psi} = \frac{\bar{\omega} - i\bar{y}}{\sqrt{2}} \equiv :e^{-i\sqrt{2}X_R}:, \quad (\text{A.2})$$

where the periodicity of  $X$  is  $2\pi R_0$ , with  $R_0 = 1/\sqrt{2}$ . These systems lead to the same  $U(1)$  left- and right-moving current algebras generated by

$$J_L = :\psi\psi^*: \equiv i\sqrt{2}\partial X, \quad J_R = :\bar{\psi}\bar{\psi}^*: \equiv -i\sqrt{2}\bar{\partial} X. \quad (\text{A.3})$$

In general, a non-left/right-symmetric model involves sectors characterized by specific boundary conditions of the complex fermions  $\psi, \bar{\psi}$  on the genus one Riemann surface,

$$\begin{aligned} \psi(z+1) &= e^{i\pi(\gamma+h_L)} \psi(z), & \bar{\psi}(\bar{z}+1) &= e^{-i\pi(\gamma+h_R)} \bar{\psi}(\bar{z}), \\ \psi(z+\tau) &= e^{i\pi(\delta+g_L)} \psi(z), & \bar{\psi}(\bar{z}+\bar{\tau}) &= e^{-i\pi(\delta+g_R)} \bar{\psi}(\bar{z}), \end{aligned} \quad (\text{A.4})$$

whose bosonic counterpart for the chiral bosons  $X_{L,R}$  leads

$$\begin{aligned} X_L(z+1) &= X_L(z) + \pi R_0(\gamma + h_L), & X_R(\bar{z}+1) &= X_R(\bar{z}) + \pi R_0(\gamma + h_R), \\ X_L(z+\tau) &= X_L(z) + \pi R_0(\delta + g_L), & X_R(\bar{z}+\bar{\tau}) &= X_R(\bar{z}) + \pi R_0(\delta + g_R). \end{aligned} \quad (\text{A.5})$$

In the above notations, which are chosen for later convenience,  $\gamma$  and  $\delta$  are integers, while  $h_L$ ,  $g_L$  and  $h_R$ ,  $g_R$  are real constants referred as left-moving and right-moving shifts.

The model can also involve a  $\mathbb{Z}_2$  twist action on the bosonic coordinate,  $X \rightarrow -X$ , whose translation in fermionic language is  $(\omega, y, \bar{\omega}, \bar{y}) \rightarrow (\omega, -y, \bar{\omega}, -\bar{y})$ , i.e.

$$\psi \rightarrow \psi^*, \quad \bar{\psi} \rightarrow \bar{\psi}^* \quad \Longleftrightarrow \quad X \rightarrow -X. \quad (\text{A.6})$$

In this case, four sectors labeled by pairs  $(H, G)$  of integers arise, as dictated by the boundary conditions

$$\begin{aligned} J_L(z+1) &= (-1)^H J_L(z), & J_R(\bar{z}+1) &= (-1)^H J_R(\bar{z}), \\ J_L(z+\tau) &= (-1)^G J_L(z), & J_R(\bar{z}+\bar{\tau}) &= (-1)^G J_R(\bar{z}). \end{aligned} \quad (\text{A.7})$$

The marginal deformation we want to consider is implemented by the addition of the operator  $\lambda J_L J_R$ , known as Thirring interaction in fermionic language, in the worldsheet Lagrangian density. Its effect in the bosonic picture is clear, since  $\lambda J_L J_R = 2\lambda \partial X \bar{\partial} X$ , which corresponds to a change of circle squared radius,  $R_0^2 \rightarrow R^2 = R_0^2(1 + 2\lambda)$ . In the fermionic picture, the Thirring interactions can be totally absorbed by changing the boundary conditions of the worldsheet complex fermions  $\psi, \bar{\psi}$ .

#### A.1.1. Untwisted sector

In the present context, we refer as “untwisted” the sector where  $J_L$  and  $J_R$  are periodic along both directions of the worldsheet torus,  $(H, G) = (0, 0)$ . The contribution to the one-loop partition function of the complex left-moving and right-moving fermions  $\psi, \bar{\psi}$  is expressed in terms of  $\theta$ -functions according to the boundary conditions (A.4),

$$\frac{\theta\left[\begin{smallmatrix} \gamma+h_L \\ \delta+g_L \end{smallmatrix} \right] \bar{\theta}\left[\begin{smallmatrix} \gamma+h_R \\ \delta+g_R \end{smallmatrix} \right]}{\eta \bar{\eta}} \equiv e^{i\pi \frac{h'}{2}(\delta+\hat{g}')} Z_{1,1}\left[\begin{smallmatrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{smallmatrix} \middle| 0 \right](R_0), \quad (\text{A.8})$$

where the r.h.s. expresses the result in the bosonic picture, which is valid at the fermionic point  $R_0$ . The bosonic side involves naturally

$$(h', g') = (h_L - h_R, g_L - g_R), \quad (\hat{h}', \hat{g}') = \left( \frac{h_L + h_R}{2}, \frac{g_L + g_R}{2} \right), \quad (\text{A.9})$$

in terms of which we have for arbitrary radius  $R$ ,

$$Z_{1,1}\left[\begin{smallmatrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{smallmatrix} \middle| 0 \right](R) = \frac{R}{\sqrt{\tau_2} \eta \bar{\eta}} \sum_{\tilde{m}, n} e^{-\frac{\pi R^2}{\tau_2} \left| (\tilde{m} - \frac{g'}{2}) + (n - \frac{h'}{2}) \tau \right|^2 + i\pi (\tilde{m} \hat{h}' - n \hat{g}')} e^{i\pi (\tilde{m} n + \tilde{m} \gamma - n \delta)}. \quad (\text{A.10})$$

The identity (A.8) can be derived by writing the powers of  $q$  and  $\bar{q}$  in the  $\theta$ -functions as  $q^{\frac{1}{2}(m - \frac{\gamma+h_L}{2})^2} \bar{q}^{\frac{1}{2}(m - n - \frac{\gamma+h_R}{2})^2}$  and performing a Poisson resummation on the momentum charge  $m$  [36]. The phase  $e^{i\pi \frac{h'}{2}(\delta+\hat{g}')}$  expresses the non-trivial behavior of the  $(1, 1)$ -conformal block under modular transformation, while  $Z_{1,1}$  is modular covariant. Actually,  $Z_{1,1}$  couples the modular covariant  $\Gamma_{1,1}$ -lattice shifted by  $(h', g')$  and  $(\hat{h}', \hat{g}')$ ,

$$\Gamma_{1,1}\left[\begin{smallmatrix} h' \\ g' \end{smallmatrix} \middle| \begin{smallmatrix} \hat{h}' \\ \hat{g}' \end{smallmatrix} \right](R) = \frac{R}{\sqrt{\tau_2}} \sum_{\tilde{m}, n} e^{-\frac{\pi R^2}{\tau_2} \left| (\tilde{m} - \frac{g'}{2}) + (n - \frac{h'}{2}) \tau \right|^2 + i\pi (\tilde{m} \hat{h}' - n \hat{g}')}}, \quad (\text{A.11})$$

to the characters  $(\gamma, \delta)$  via the modular invariant phase  $e^{i\pi (\tilde{m} n + \tilde{m} \gamma - n \delta)}$ . The modular transformations act as

$$\begin{aligned}
\tau &\rightarrow -\frac{1}{\tau} \iff (h', g') \rightarrow (h', g')S, \quad (\hat{h}', \hat{g}') \rightarrow (\hat{h}', \hat{g}')S, \quad (\gamma, \delta) \rightarrow (\gamma, \delta)S, \\
\tau &\rightarrow \tau + 1 \iff (h', g') \rightarrow (h', g')\mathcal{T}, \quad (\hat{h}', \hat{g}') \rightarrow (\hat{h}', \hat{g}')\mathcal{T}, \quad (\gamma, \delta) \rightarrow (\gamma, \delta + \gamma - 1), \\
&\text{where } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{A.12}$$

Given the fact that the marginal deformation by  $J_L J_R$  in the bosonic picture amounts to changing the argument  $R$  of  $Z_{1,1}$ , whose modular properties are  $R$ -independent, the contribution to the partition function of the *untwisted sector of the  $R$  modulus-deformed fermionic construction* is obtained by replacing

$$\frac{\theta[\frac{\gamma+h_L}{\delta+g_L}]\bar{\theta}[\frac{\gamma+h_R}{\delta+g_R}]}{\eta\bar{\eta}} \longrightarrow e^{i\pi\frac{h'_L}{2}(\delta+\hat{g}')} Z_{1,1}\left[\begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| 0 \right](R). \tag{A.13}$$

Note that the particular values  $R = \frac{p}{q}R_0$  for  $\frac{p}{q}$  rational can be realized in fermionic language by implementing a  $\mathbb{Z}_p \times \mathbb{Z}_q$  quotient on the theory, where the orbifold generators act as phases similar to Eq. (A.4), or shifts similar to Eq. (A.5) in bosonic language. A well known example of this procedure is that the left/right-symmetric compactification on  $S^1(R)/\mathbb{Z}_2^{\text{shift}}$  is equivalent to that on  $S^1(R/2)$ .

#### A.1.2. Twisted sectors

The twisted sectors, which have  $H, G$  not both even, can be considered in the bosonic language for arbitrary radius  $R$ . The boundary conditions (A.7) imply  $\partial X$  and  $\bar{\partial} X$  have vanishing constant modes, so that no  $R$ -dependent zero mode lattice arises in these sectors and the  $J_L J_R$  marginal deformation is trivial. The alternative point of view, where the switch from  $R_0$  to  $R = \frac{p}{q}R_0$  is implemented in the fermionic construction by a  $\mathbb{Z}_p \times \mathbb{Z}_q$  orbifold action, leads to the same conclusion. For instance, when  $H = 1$ , the key point is that the boundary conditions for some phases  $\varphi_L, \varphi_R$  are

$$\psi(z+1) = (e^{i\varphi_L}\psi)^*(z), \quad \bar{\psi}(\bar{z}+1) = (e^{i\varphi_R}\bar{\psi})^*(\bar{z}), \tag{A.14}$$

and become trivial under the redefinitions

$$\tilde{\psi}(z) \equiv e^{\frac{i}{2}\varphi_L}\psi(z), \quad \tilde{\bar{\psi}}(\bar{z}) \equiv e^{\frac{i}{2}\varphi_R}\bar{\psi}(\bar{z}). \tag{A.15}$$

In other words, *the twisted sectors of the  $R$  modulus-deformed fermionic construction are those of the undeformed one.*

In a twisted sector, the boundary conditions of  $\omega, y, \bar{\omega}, \bar{y}$  along the cycles of the genus one Riemann surface are either periodic or antiperiodic. In other words, when a  $\mathbb{Z}_2$  twist is implemented,  $h_{L,R}$  and  $g_{L,R}$  are restricted to be integer. The contribution of  $\omega, y, \bar{\omega}, \bar{y}$  to the one-loop partition function is

$$\frac{1}{\eta\bar{\eta}} \theta^{\frac{1}{2}}[\frac{\gamma+h_L}{\delta+g_L}] \theta^{\frac{1}{2}}[\frac{\gamma+h_L+H}{\delta+g_L+G}] \bar{\theta}^{\frac{1}{2}}[\frac{\gamma+h_R}{\delta+g_R}] \bar{\theta}^{\frac{1}{2}}[\frac{\gamma+h_R+H}{\delta+g_R+G}] \equiv e^{i\varphi[\frac{\gamma; h_L, h_R}{\delta; g_L, g_R} \middle| \frac{H}{G}]} Z_{1,1}\left[\begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| \frac{H}{G} \right], \tag{A.16}$$

where the r.h.s. shows the result in non-left/right-symmetric orbifold language. In fact, the bosons yield

$$Z_{1,1}\left[\begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| \frac{H}{G} \right] = \left| \frac{2\eta}{\theta[1-G]} \right| \mathcal{P}\left[\begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| \frac{H}{G} \right], \tag{A.17}$$

where  $\mathcal{P}$  is a modular invariant projector that picks up the only non-trivial contributions, which arise from the fixed points of the non-symmetric  $\mathbb{Z}_2$  orbifold,

$$\mathcal{P} \left[ \begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| \begin{matrix} H \\ G \end{matrix} \right] = \frac{1}{2} \left( 1 + e^{i\pi(\gamma+h_L)(\delta+g_L)} \right) \frac{1}{2} \left( 1 + e^{i\pi(\gamma+h_L+H)(\delta+g_L+G)} \right) \\ \times \frac{1}{2} \left( 1 + e^{i\pi(\gamma+h_R)(\delta+g_R)} \right) \frac{1}{2} \left( 1 + e^{i\pi(\gamma+h_R+H)(\delta+g_R+G)} \right). \quad (\text{A.18})$$

Beside Eq. (A.12), the modular transformations act on  $(H, G)$  as,

$$\tau \rightarrow -\frac{1}{\tau} \iff (H, G) \rightarrow (H, G)S, \quad \tau \rightarrow \tau + 1 \iff (H, G) \rightarrow (H, G)\mathcal{T}. \quad (\text{A.19})$$

The relation (A.16) is obtained via the  $\theta$ -function identities

$$\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\eta^3, \quad \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \quad \text{i.e.} \quad \theta_2 \theta_3 \theta_4 = 2\eta^3, \quad \theta_1 = 0, \quad (\text{A.20})$$

while from the fermionic point of view, the projector  $\mathcal{P}$  captures the fact that the sectors that involve  $\theta_1$  are vanishing. The phase in the r.h.s. of Eq. (A.16) is

$$\varphi \left[ \begin{matrix} \gamma; h_L, h_R \\ \delta; g_L, g_R \end{matrix} \middle| \begin{matrix} H \\ G \end{matrix} \right] = \frac{\pi}{2} (g_L - g_R)(1 - H - G) \quad \text{for} \quad \delta + g_L, \delta + g_R, H, G \in \{0, 1\}, \quad (\text{A.21})$$

but varies accordingly, when some of the above arguments take other integer values.

### A.1.3. Left/right-symmetric case

At this stage, the left- and right-moving shifts we have described are the most general ones. In the following, we concentrate on a case of particular interest that corresponds to the left/right-symmetric bosonic compactification.

In sectors where  $\hat{h}'$  and  $\hat{g}'$  vanish, we define

$$(h, g) := (h_R, g_R) = (-h_L, -g_L) \quad \left( \text{i.e. } (h, g) = \left( -\frac{h'}{2}, -\frac{g'}{2} \right) \text{ and } (\hat{h}', \hat{g}') = (0, 0) \right), \quad (\text{A.22})$$

and consider the fermionic block

$$e^{i\pi(k-\frac{1}{2})(hG-gH)} \left( e^{i\pi h\delta} \frac{\theta \begin{bmatrix} \gamma-h \\ \delta-g \end{bmatrix} \bar{\theta} \begin{bmatrix} \gamma+h \\ \delta+g \end{bmatrix}}{\eta \bar{\eta}} \right)^{\frac{1}{2}} \left( e^{i\pi h(\delta+G)} \frac{\theta \begin{bmatrix} \gamma+H-h \\ \delta+G-g \end{bmatrix} \bar{\theta} \begin{bmatrix} \gamma+H+h \\ \delta+G+g \end{bmatrix}}{\eta \bar{\eta}} \right)^{\frac{1}{2}}. \quad (\text{A.23})$$

Since the quantity  $hG - gH$  is modular invariant, the phase  $e^{i\pi(k-\frac{1}{2})(hG-gH)}$  can be introduced for any real  $k$ . Moreover, we see from Eq. (A.8) that the specific insertion of phase  $e^{i\pi h(\delta+\frac{G}{2})}$  makes the fermionic block modular covariant and allows  $\gamma, \delta$  to be defined modulo 2. Summing over  $\gamma, \delta$  equal to 0, 1, we obtain when  $h, g$  are restricted to be integer,

$$Z_{1,1}^{\text{fer},k} \left[ \begin{matrix} h \\ g \end{matrix} \middle| \begin{matrix} H \\ G \end{matrix} \right] := e^{i\pi(k-\frac{1}{2})(hG-gH)} \frac{1}{2} \sum_{\gamma, \delta} e^{i\pi[h(\delta+\frac{G}{2})-g(\gamma+h+\frac{H}{2})]} \left| \frac{\theta \begin{bmatrix} \gamma+h \\ \delta+g \end{bmatrix}}{\eta} \right| \left| \frac{\theta \begin{bmatrix} \gamma+H+h \\ \delta+G+g \end{bmatrix}}{\eta} \right| \\ = e^{ik\pi(hG-gH)} \frac{1}{2} \sum_{\tilde{\gamma}, \tilde{\delta}} e^{i\pi(-g\tilde{\gamma}+h\tilde{\delta}-hg)} \left| \frac{\theta \begin{bmatrix} \tilde{\gamma} \\ \tilde{\delta} \end{bmatrix}}{\eta} \right| \left| \frac{\theta \begin{bmatrix} \tilde{\gamma}+H \\ \tilde{\delta}+G \end{bmatrix}}{\eta} \right|, \quad (\text{A.24})$$



where we have defined  $\tilde{\gamma} = \gamma + h$  and  $\tilde{\delta} = \delta + g$  in the second line. From now on, we restrict  $k$  to be integer modulo 2, so that  $h, g$  and  $H, G$  are defined modulo 2 in the above expression. In this case, we also have

$$Z_{1,1}^{\text{fer},k} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] = e^{i(1-k)\pi(hG-gH)} \frac{1}{2} \sum_{\tilde{\gamma}, \tilde{\delta}} e^{i\pi(-g\tilde{\gamma}+h\tilde{\delta}-hg)} \left| \frac{\theta[\frac{\tilde{\gamma}+H}{\tilde{\delta}+G}]}{\eta} \right| \left| \frac{\theta[\frac{\tilde{\gamma}}{\tilde{\delta}}]}{\eta} \right|, \quad (\text{A.25})$$

which shows that changing  $k \rightarrow 1 - k$  corresponds to imposing the twist to act on  $\omega, \bar{\omega}$  instead of  $y, \bar{y}$ , which leads to an equivalent model.

For  $(H, G) = (0, 0)$ , we obtain from the definition (A.23)

$$\begin{aligned} Z_{1,1}^{\text{fer},k} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] &= \frac{1}{2} \sum_{\gamma, \delta} e^{i\pi h\delta} \frac{\theta[\frac{\gamma-h}{\delta-g}] \bar{\theta}[\frac{\gamma+h}{\delta+g}]}{\eta \bar{\eta}} \\ &= \frac{1}{2} \sum_{\gamma, \delta} \frac{R_0}{\sqrt{\tau_2} \eta \bar{\eta}} \sum_{\tilde{m}, n} e^{-\frac{\pi R_0^2}{\tau_2} |(\tilde{m}+g)+(n+h)\tau|^2} e^{i\pi(\tilde{m}n+\tilde{m}\gamma-n\delta)} \\ &= \frac{2R_0}{\sqrt{\tau_2} \eta \bar{\eta}} \sum_{\tilde{m}', n'} e^{-\frac{\pi(2R_0)^2}{\tau_2} |(\tilde{m}'+\frac{g}{2})+(n'+\frac{h}{2})\tau|^2} := \frac{\Gamma_{1,1}[\frac{h}{g}](2R_0)}{\eta \bar{\eta}}, \end{aligned} \quad (\text{A.26})$$

where the sum over  $\gamma, \delta$  projects out the odd values of  $\tilde{m}$  and  $n$ . Thus, we recover the well known bosonic  $\Gamma_{1,1}$ -lattice considered in Eq. (3.5), with shifts  $(h, g)$  and radius  $R_1 = 2R_0$ .

For  $(H, G) \neq (0, 0)$  modulo 2, we use Eqs. (A.16)–(A.18) applied for  $h_{L,R} = g_{L,R} = 0$  to write

$$\begin{aligned} Z_{1,1}^{\text{fer},k} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] &= e^{ik\pi(hG-gH)} \frac{1}{2} \sum_{\gamma, \delta} e^{i\pi(-g\gamma+h\delta-hg)} \left| \frac{2\eta}{\theta[\frac{1-H}{1-G}]} \right| \delta_{\gamma\delta, 0 \bmod 2} \delta_{(\gamma+H)(\delta+G), 0 \bmod 2} \\ &= e^{ik\pi(hG-gH)} \left| \frac{2\eta}{\theta[\frac{1-H}{1-G}]} \right| \left( \delta_{(h,g), (0,0) \bmod 2} + \delta_{(h,g), (H,G) \bmod 2} \right) \\ &= \left| \frac{2\eta}{\theta[\frac{1-H}{1-G}]} \right| \delta_{\left| \begin{smallmatrix} h \\ g \end{smallmatrix} \right| \begin{smallmatrix} H \\ G \end{smallmatrix} \right|, 0 \bmod 2}, \end{aligned} \quad (\text{A.27})$$

which is nothing but the  $(H, G)$ -twisted and  $(h, g)$ -shifted sector of a circle compactification [15].

Using the rule shown in Eq. (A.13), the  $(1, 1)$ -block of the  $R$ -modulus deformed fermionic construction that realizes the left/right-symmetric case in bosonic language is obtained by substituting

$$\begin{aligned} Z_{1,1}^{\text{fer},k} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] &= e^{ik\pi(hG-gH)} \frac{1}{2} \sum_{\gamma, \delta} e^{i\pi(-g\gamma+h\delta-hg)} \left| \frac{\theta[\frac{\gamma}{\delta}]}{\eta} \right| \left| \frac{\theta[\frac{\gamma+H}{\delta+G}]}{\eta} \right| \longrightarrow \\ &Z_{1,1} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] (2R), \end{aligned} \quad (\text{A.28})$$

where the r.h.s. is the block associated to a twisted and shifted circle compactification at arbitrary radius  $2R$ ,

$$Z_{1,1}\left[\begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix}\right](2R) = \begin{cases} \frac{\Gamma_{1,1}[\begin{smallmatrix} h \\ g \end{smallmatrix}](2R)}{\eta\bar{\eta}}, & \text{when } (H, G) = (0, 0) \bmod 2 \\ \left| \frac{2\eta}{\theta[\begin{smallmatrix} 1-H \\ 1-G \end{smallmatrix}]} \right| \delta_{\begin{smallmatrix} h \\ g \end{smallmatrix} \mid \begin{smallmatrix} H \\ G \end{smallmatrix}, 0 \bmod 2}, & \text{when } (H, G) \neq (0, 0) \bmod 2. \end{cases} \quad (\text{A.29})$$

Before considering the two coordinates compactification, we would like to make some remarks. Summing over the shifts  $h, g$ , we obtain

$$\frac{1}{2} \sum_{h,g} Z_{1,1}^{\text{fer},k} \left[ \begin{smallmatrix} h \\ g \end{smallmatrix} \middle| \begin{smallmatrix} H \\ G \end{smallmatrix} \right] = \frac{1}{2} \sum_{\gamma,\delta} e^{-i\pi(\gamma+kH)(\delta+kG)} \left| \frac{\theta[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]}{\eta} \right| \left| \frac{\theta[\begin{smallmatrix} \gamma+H \\ \delta+G \end{smallmatrix}]}{\eta} \right| = Z_{1,1}[\begin{smallmatrix} H \\ G \end{smallmatrix}](R_0), \quad (\text{A.30})$$

where

$$Z_{1,1}[\begin{smallmatrix} H \\ G \end{smallmatrix}](R) = \begin{cases} \frac{\Gamma_{1,1}(R)}{\eta\bar{\eta}}, & \text{when } (H, G) = (0, 0) \bmod 2 \\ \left| \frac{2\eta}{\theta[\begin{smallmatrix} 1-H \\ 1-G \end{smallmatrix}]} \right|, & \text{when } (H, G) \neq (0, 0) \bmod 2 \end{cases} \quad (\text{A.31})$$

and  $\Gamma_{1,1}(R) \equiv \Gamma_{1,1}[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](R)$  is the circle compactification lattice. Eq. (A.30) expresses the geometrical fact that

$$\frac{S^1(2R_0)}{\mathbb{Z}_2^{\text{shift}} \times \mathbb{Z}_2} = \frac{S^1(R_0)}{\mathbb{Z}_2}, \quad (\text{A.32})$$

i.e. that the shift divides the radius of compactification by a factor of 2, even when the circle is twisted. However, from the fermionic point of view, the natural definition of the twisted  $(1, 1)$ -conformal block is without the phase  $e^{-i\pi(\gamma+kH)(\delta+kG)}$  present in Eq. (A.30). Thus, we take

$$Z_{1,1}^{\text{fer}}[\begin{smallmatrix} H \\ G \end{smallmatrix}] := \frac{1}{2} \sum_{\gamma,\delta} \left| \frac{\theta[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]}{\eta} \right| \left| \frac{\theta[\begin{smallmatrix} \gamma+H \\ \delta+G \end{smallmatrix}]}{\eta} \right| = \begin{cases} \frac{1}{2} \sum_{\gamma,\delta} \frac{\theta[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]\bar{\theta}[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]}{\eta\bar{\eta}}, & \text{when } (H, G) = (0, 0) \bmod 2 \\ \left| \frac{2\eta}{\theta[\begin{smallmatrix} 1-H \\ 1-G \end{smallmatrix}]} \right|, & \text{when } (H, G) \neq (0, 0) \bmod 2, \end{cases} \quad (\text{A.33})$$

where we have used Eqs. (A.16)–(A.18) for  $h_{L,R} = g_{L,R} = 0$  in the second line. Since Eq. (A.8) gives

$$\begin{aligned} \frac{1}{2} \sum_{\gamma,\delta} \frac{\theta[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]\bar{\theta}[\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}]}{\eta\bar{\eta}} &= \frac{R_0}{\sqrt{\tau_2}\eta\bar{\eta}} \sum_{\tilde{m},n} e^{-\frac{\pi R_0^2}{\tau_2}|\tilde{m}+n\tau|^2} \frac{1}{2} \sum_{\gamma,\delta} e^{i\pi(\tilde{m}n+\tilde{m}\gamma-n\delta)} \\ &= \frac{2R_0}{\sqrt{\tau_2}\eta\bar{\eta}} \sum_{\tilde{m}',n'} e^{-\frac{\pi(2R_0)^2}{\tau_2}|\tilde{m}'+n'\tau|^2} = \frac{\Gamma(2R_0)}{\eta\bar{\eta}}, \end{aligned} \quad (\text{A.34})$$

we finally conclude as expected that

$$Z_{1,1}^{\text{fer}}[\begin{smallmatrix} H \\ G \end{smallmatrix}] = Z_{1,1}[\begin{smallmatrix} H \\ G \end{smallmatrix}](2R_0). \quad (\text{A.35})$$

Comparing Eqs. (A.30), (A.33) and (A.35), we see that if the shift divides the radius by 2 in bosonic language, it flips the signs in front of  $\theta_1$ -functions in fermionic language. We thus have

$Z_{1,1}[\frac{H}{G}](2R_0) = Z_{1,1}[\frac{H}{G}](R_0)$ , a fact that can be understood as a T-duality. Actually, since  $2R_0 = 1/R_0$ , the shift operation that changes  $2R_0 \rightarrow R_0$  is equivalent to the operation  $1/R_0 \rightarrow R_0$ .

Before concluding this subsection, we would like to mention that in order to simplify formulas in the core of our paper, we have used the convention to take  $Z_{1,1}[\frac{h}{g}|\frac{H}{G}](R)$  rather than  $Z_{1,1}[\frac{h}{g}|\frac{H}{G}](2R)$  in the r.h.s. of the substitution (A.28).

## A.2. Two coordinates compactification

Proceeding in a similar way for a second coordinate, we can deform even further an initial fermionic model by switching on the full metric  $G_{ij}$  and antisymmetric tensor  $B_{ij}$  moduli,  $i = 1, 2$ . This is done without changing the modular properties of the initial model constructed at the fermionic point. As before, we introduce integers  $\gamma^i, \delta^i$ , together with real left- and right-moving shifts  $h_{L,R}^i, g_{L,R}^i$ . In case a  $\mathbb{Z}_2$  twist is implemented, we suppose it acts simultaneously on the two coordinates.

### A.2.1. Untwisted sector

We start with the sector  $(H, G) = (0, 0)$ . Defining linear combinations  $h'^i, g'^i, \hat{h}'^i, \hat{g}'^i$  as in Eq. (A.9), the undeformed  $(2, 2)$ -conformal block takes the form

$$\begin{aligned} & \frac{\theta[\frac{\gamma^1+h_L^1}{\delta^1+g_L^1}]\bar{\theta}[\frac{\gamma^1+h_R^1}{\delta^1+g_R^1}]}{\eta\bar{\eta}} \frac{\theta[\frac{\gamma^2+h_L^2}{\delta^2+g_L^2}]\bar{\theta}[\frac{\gamma^2+h_R^2}{\delta^2+g_R^2}]}{\eta\bar{\eta}} \\ & \equiv e^{i\pi[\frac{h^1}{2}(\delta^1+\hat{g}'^1)+\frac{h^2}{2}(\delta^2+\hat{g}'^2)]} Z_{2,2}\left[\begin{smallmatrix} \gamma^i; h_L^i, h_R^i \\ \delta^i; g_L^i, g_R^i \end{smallmatrix} \middle| 0 \right](T_0, U_0), \end{aligned} \quad (\text{A.36})$$

where  $(T_0, U_0) = (\frac{i}{2}, i)$  and, for arbitrary  $T$  and  $U$ ,

$$\begin{aligned} & Z_{2,2}\left[\begin{smallmatrix} \gamma^i; h_L^i, h_R^i \\ \delta^i; g_L^i, g_R^i \end{smallmatrix} \middle| 0 \right](T, U) \\ & = \frac{\sqrt{\det G}}{\tau_2(\eta\bar{\eta})^2} \sum_{\tilde{m}^i, n^i} e^{-\frac{\pi}{\tau_2}[\tilde{m}^i - \frac{g^i}{2} + (n^i - \frac{h^i}{2})\tau](G_{ij}+B_{ij})[\tilde{m}^j - \frac{g^j}{2} + (n^j - \frac{h^j}{2})\bar{\tau}]} \\ & \quad \times e^{i\pi(\tilde{m}_i \hat{h}^i - n_i \hat{g}^i)} e^{i\pi(\tilde{m}_i n^i + \tilde{m}_i \gamma^i - n_i \delta^i)}, \end{aligned} \quad (\text{A.37})$$

with  $T, U$  related to the metric and antisymmetric tensor as

$$G_{ij} = \frac{\text{Im } T}{\text{Im } U} \begin{pmatrix} 1 & \text{Re } U \\ \text{Re } U & |U|^2 \end{pmatrix}, \quad B_{ij} = \text{Re } T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.38})$$

Here also,  $Z_{2,2}$  couples non-trivially the  $\Gamma_{2,2}$ -lattice shifted by  $(h^i, g^i)$  and  $(\hat{h}^i, \hat{g}^i)$ ,

$$\frac{\sqrt{\det G}}{\tau_2} \sum_{\tilde{m}^i, n^i} e^{-\frac{\pi}{\tau_2}[\tilde{m}^i - \frac{g^i}{2} + (n^i - \frac{h^i}{2})\tau](G_{ij}+B_{ij})[\tilde{m}^j - \frac{g^j}{2} + (n^j - \frac{h^j}{2})\bar{\tau}]} e^{i\pi(\tilde{m}_i \hat{h}^i - n_i \hat{g}^i)} \quad (\text{A.39})$$

to the characters  $(\gamma^i, \delta^i)$ , via the modular invariant phase  $e^{i\pi(\tilde{m}_i n^i + \tilde{m}_i \gamma^i - n_i \delta^i)}$ .

The contribution to the partition function of the *untwisted sector of the  $T, U$  moduli-deformed fermionic construction* is obtained by replacing  $T_0, U_0$  by arbitrary  $T$  and  $U$ :

$$\frac{\theta[\gamma^1+h_L^1]\bar{\theta}[\gamma^1+h_R^1]}{\eta\bar{\eta}} \frac{\theta[\gamma^2+h_L^2]\bar{\theta}[\gamma^2+h_R^2]}{\eta\bar{\eta}} \longrightarrow e^{i\pi\left[\frac{h^1}{2}(\delta^1+\hat{g}^1)+\frac{h^2}{2}(\delta^2+\hat{g}^2)\right]} Z_{2,2}\left[\begin{smallmatrix}\gamma^i; h_L^i, h_R^i \\ \delta^i; g_L^i, g_R^i\end{smallmatrix}\middle|0\right](T, U). \quad (\text{A.40})$$

### A.2.2. Twisted sectors

When  $H$  and  $G$  are not both even, the associated *twisted sectors of the  $T, U$  moduli-deformed fermionic construction are those of the undeformed one*. This is again due to the fact that they are moduli-independent, which implies that the expressions of their conformal blocks are those given at the fermionic point  $(T_0, U_0)$ :

$$\frac{1}{(\eta\bar{\eta})^2} \prod_i \left( \theta^{\frac{1}{2}}[\gamma^i+h_L^i]\theta^{\frac{1}{2}}[\gamma^i+h_L^i+H]\bar{\theta}^{\frac{1}{2}}[\gamma^i+h_R^i]\bar{\theta}^{\frac{1}{2}}[\gamma^i+h_R^i+H] \right) \\ \equiv e^{i\sum_i \varphi\left[\begin{smallmatrix}\gamma^i; h_L^i, h_R^i \\ \delta^i; g_L^i, g_R^i\end{smallmatrix}\middle|H\right]} Z_{2,2}\left[\begin{smallmatrix}\gamma^j; h_L^j, h_R^j \\ \delta^j; g_L^j, g_R^j\end{smallmatrix}\middle|H\right], \quad (\text{A.41})$$

where in bosonic language we have

$$Z_{2,2}\left[\begin{smallmatrix}\gamma^j; h_L^j, h_R^j \\ \delta^j; g_L^j, g_R^j\end{smallmatrix}\middle|H\right] = \left| \frac{2\eta}{\theta[1-H]} \right|^2 \prod_i \mathcal{P}\left[\begin{smallmatrix}\gamma^i; h_L^i, h_R^i \\ \delta^i; g_L^i, g_R^i\end{smallmatrix}\middle|H\right]. \quad (\text{A.42})$$

### A.2.3. Left/right-symmetric case

Defining shifts  $h^i, g^i$  as in Eq. (A.22), we consider for integer  $k^i$ 's the fermionic conformal block

$$\prod_i e^{i\pi(k^i-\frac{1}{2})(h^i G - g^i H)} \left( e^{i\pi h^i \delta^i} \frac{\theta[\gamma^i-h^i]\bar{\theta}[\gamma^i+h^i]}{\eta\bar{\eta}} \right)^{\frac{1}{2}} \\ \times \left( e^{i\pi h^i(\delta^i+G)} \frac{\theta[\gamma^i+H-h^i]\bar{\theta}[\gamma^i+H+h^i]}{\eta\bar{\eta}} \right)^{\frac{1}{2}}, \quad (\text{A.43})$$

where  $\gamma^i, \delta^i$  are integer modulo 2. Proceeding as in the one coordinate case, we sum over  $\gamma^i, \delta^i$  and find, when  $h^i, g^i$  are integer,

$$Z_{2,2}^{\text{fer},k^j}\left[\begin{smallmatrix}h^1, h^2 \\ g^1, g^2\end{smallmatrix}\middle|H\right] := \prod_i e^{ik^i\pi(h^i G - g^i H)} \frac{1}{2} \sum_{\gamma^i, \delta^i} e^{i\pi(-g^i \gamma^i + h^i \delta^i - h^i g^i)} \left| \frac{\theta[\gamma^i]}{\eta} \right| \left| \frac{\theta[\gamma^i+H]}{\eta} \right| \\ = \begin{cases} \frac{\Gamma_{2,2}[\begin{smallmatrix}h^1, h^2 \\ g^1, g^2\end{smallmatrix}](4T_0, U_0)}{(\eta\bar{\eta})^2}, & \text{when } (H, G) = (0, 0) \bmod 2 \\ \frac{4\eta\bar{\eta}}{\theta[1-H]\bar{\theta}[1-H]} \delta_{\begin{smallmatrix}h^1 & H \\ g^1 & G\end{smallmatrix}, 0 \bmod 2} \delta_{\begin{smallmatrix}h^2 & H \\ g^2 & G\end{smallmatrix}, 0 \bmod 2}, & \text{when } (H, G) \neq (0, 0) \bmod 2, \end{cases} \quad (\text{A.44})$$

where the  $\Gamma_{2,2}$  shifted lattice is defined in Eq. (4.2).

As said in Eq. (A.40), the moduli deformation amounts to changing the argument of the lattice as  $(4T_0, U_0) \rightarrow (4T, U)$ , where  $T, U$  are arbitrary. However, in the core of the paper, we found convenient to take the lattice argument at arbitrary point in moduli space to be  $(T, U)$ , as indicated in Eq. (6.1).

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